DYNAMIC RISK MEASURING AND PRICING IN INCOMPLETE MARKETS

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PLAN OF MY TALK

Plan

INTRODUCTION

MONETARY RISK MEASURES:

Coherent risk measures: Artzner, Delbaen, Eber, Heath (1999)

Convex risk measure: Föllmer, Schied (2002)and Frittelli, Rosazza Gianin (2002)

CONDITIONAL RISK MEASURES

on a probability space : Detlefsen and Scandolo (2005)

in a context of uncertainty : Bion-Nadal (2004)

DYNAMIC RISK MEASURES

- Coherent Dynamic Risk Measuring: Delbaen (2003) and Artzner, Delbaen, Eber, Heat, Ku (2004), and Riedel (2004)
- Convex dynamic risk measuring considered in: Frittelli and Rosaza Gianin (2004), Klöppel, Schweizer (2005), Cheredito, Delbaen, Kupper (2006), Bion-Nadal (2006), Föllmer and Penner (2006)
- g expectations or Backward Stochastic Differential Equations : Peng (2004), Rosazza Gianin (2004) and Barrieu El Karoui (2005)

PLAN

Conditional Risk Measure

Consider a probability space $(\Omega, \mathcal{G}, \mathcal{P})$ and $\mathcal{F}$ a subsigma-algebra of $\mathcal{G}$.

Definition

A mapping

$$\rho : L^\infty(\Omega, \mathcal{G}, P) \to L^\infty(\Omega, \mathcal{F}, P)$$

is a convex risk measure on $(\Omega, \mathcal{G}, \mathcal{P})$ conditional to the probability space $(\Omega, \mathcal{F}, \mathcal{P})$ if it satisfies the following conditions:

1. monotonicity: for all X,$Y \in L^\infty(\Omega, \mathcal{G}, P)$ if $X \leq Y$ then $\rho(Y) \leq \rho(X)$ $\;P\ a.s.$
2. translation invariance: for all $Y \in L^\infty(\Omega, \mathcal{F}, P)$, for all $X \in \mathcal{X}$,
$$\rho(X+Y) = \rho(X) - Y$$
3. convexity : for all X,$Y \in L^\infty(\Omega, \mathcal{G}, P)$ $\forall \lambda \in [0, 1]$, $\rho(\lambda X + (1 - \lambda Y) \leq \lambda\rho(X) + (1 - \lambda)\rho(Y)$

Conditional Risk Measure

Definition

A convex conditional risk measure ρ is continuous from above (resp below) if for every decreasing (resp increasing) sequence X_n of elements of $L^\infty(\Omega, \mathcal{G}, P)$ converging to X, the increasing (resp decreasing) sequence $\rho(X_n)$ converges to $\rho(X)$

Recall that every convex risk measure on $L^\infty(\Omega, \mathcal{G}, P)$ conditional to $L^\infty(\Omega, \mathcal{F}, P)$ continuous from above has a representation of the kind:

Proposition

$$\forall X \in L^\infty(\mathcal{G}) \;\; \rho(X) = ess\ sup_{Q \in \tilde{\mathcal{M}}}((E_Q(-X|\mathcal{F}) - \alpha^m(Q)) \tag{1}$$

where $\tilde{\mathcal{M}} = \{Q \; on \; (\Omega, \mathcal{G}) \; |Q \ll P \; , \; Q_{|\mathcal{F}} = P\}$.

TIME CONSISTENCY IN DISCRETE TIME

We consider a space Ω and a numerable increasing family of σ-algebras $\mathcal{F}_n$ on Ω such that $L^\infty(\Omega, \mathcal{F}_0, P) = \boldsymbol{R}$

DEFINITION

A dynamic risk measure on $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\in \boldsymbol{N}}, P)$ is a family $((\rho_{n,n+1})_{n\in \boldsymbol{N}})$ where $\rho_{n,n+1}$ is a convex risk measure on $(\Omega, \mathcal{F}_{n+1}, P)$ conditional to $(\Omega, \mathcal{F}_n, P)$.

PROPOSITION

Let $n < m$. Consider a dynamic risk measure as in the preceding definition. Then the relation $\rho_{n,m} = \rho_{n,n+1} o(-\rho_{n+1,n+2})...o(-\rho_{m-1,m})$ defines a risk measure on $(\Omega, \mathcal{F}_m, P)$ conditional to $(\Omega, \mathcal{F}_n, P)$. The family $(\rho_{n,m})$ is time-consistent, i.e. $\forall n < m < r$ $\rho_{n,r} = \rho_{n,m} o(-\rho_{m,r})$.

Remark:
The notion of time consistency first appeared in the work of Peng (2004)

TIME CONSISTENCY IN DISCRETE TIME

TIME CONSISTENCY DYNAMIC RISK MEASURE AS A SINGLE CONDITIONAL RISK MEASURE

Denote now $\tilde{\Omega} = \Omega \times N$ and $\tilde{\mathcal{F}}$ the σ-algebra generated by the sets $A_i \times \{i\}$ where $A_i \in \mathcal{F}_i$. Denote also $\tilde{\mathcal{F}}^s$ the shifted algebra generated by the sets $A_i \times \{i\}$ where $A_i \in \mathcal{F}_{i-1}$. Define the probability $\tilde{P}$ on $\tilde{\mathcal{F}}$ by:
$\tilde{P}(U_{i \in \mathbf{N}} A_i \times \{i\}) = \sum_{i \in N} \frac{1}{2^{i+1}} P(A_i)$.

PROPOSITION

There is a canonical bijection between the set of dynamic risk measures on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n \in \mathbf{N}} P)$ and the set of convex risk measures on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ conditional to $(\tilde{\Omega}, \tilde{\mathcal{F}}^s, \tilde{P})$.

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CONTINUOUS TIME

FRAMEWORK

filtered probability space $(\Omega, \mathcal{F}_\infty, (\mathcal{F}_t))_{t \in R^+}, P)$
filtration $\mathcal{F}_t$ right continuous and $\mathcal{F}_0$ is the σ-algebra generated by the P null sets of $\mathcal{F}_\infty$ thus $L^\infty(\Omega, \mathcal{F}_0, P) = \boldsymbol{R}$.
For a stopping time τ, recall that
$\mathcal{F}_\tau = \{A \in \ \mathcal{F}_\infty | \forall t \in \boldsymbol{R}^+ \ A \cap \{\tau \leq t\} \in \mathcal{F}_t\}$.

DEFINITION

1. A dynamic risk measure on $(\Omega, \mathcal{F}_\infty, (\mathcal{F}_t)_{t \in R^+}, P)$ is a family of maps $(\rho_{\sigma,\tau})_{0 \leq \sigma \leq \tau}$, ($\sigma$ and τ stopping times) defined on $L^\infty(\mathcal{F}_\tau)$ with values into $L^\infty(\mathcal{F}_\sigma)$ such that each $\rho_{\sigma,\tau}$ is a convex conditional risk measure.
2. It is time consistent if $\forall \nu \leq \sigma \leq \tau$, $\rho_{\nu,\tau} = \rho_{\nu,\sigma} o(-\rho_{\sigma,\tau})$

For characterization of time consistency in terms of the minimal penalties, the probability absolutely continuous with respect to P are crucial.

CONTINUOUS TIME

NOTATION

For every probability measure Q on $(\Omega, \mathcal{F}_\tau)$ absolutely continuous with respect to P, denote:

- $$\mathcal{A}_{i,j}(Q) = \{Y \in L^\infty(\Omega, \mathcal{F}_j, P) \mid \rho_{i,j}(Y) \leq 0 \;\; Q\ a.s.\} \tag{2}$$

- $$\begin{aligned} \alpha^m_{\sigma,\tau}(Q) &= Q\ ess\ sup_{X \in L^\infty(\Omega, \mathcal{F}_\tau, P)} (E_Q(-X|\mathcal{F}_\sigma) - \rho_{\sigma,\tau}(X)) \\ &= Q\ ess\ sup_{Y \in \mathcal{A}_{\sigma,\tau}(Q)} E_Q(-Y|\mathcal{F}_\sigma) \\ &= Q\ ess\ sup_{Y \in \mathcal{A}_{\sigma,\tau}} E_Q(-Y|\mathcal{F}_\sigma) \end{aligned} \tag{3}$$

- $\mathcal{M}^1_{\sigma,\tau}(Q) = \{R \ll P \mid R_{|\mathcal{F}_\sigma} = Q \mid E_R(\alpha^m_{\sigma,\tau}(R)) < \infty\}$

CHARACTERIZATION OF TIME CONSISTENCY

THEOREM

Consider $\rho_{\sigma,\tau}$ a dynamic risk measure continuous from above. Let $\nu \leq \sigma \leq \tau$ be stopping times. The three following conditions are equivalent:

I) *The dynamic risk measure is time consistent i.e.*

$$\rho_{\nu,\tau}(X) = \rho_{\nu,\sigma}(-\rho_{\sigma,\tau}(X)) \ \ \forall X \in L^{\infty}(\Omega, \mathcal{F}_{\tau})$$

II) *For every probability measure Q absolutely continuous with respect to P,*

$$\mathcal{A}_{\nu,\tau}(Q) = \mathcal{A}_{\nu,\sigma}(Q) + \mathcal{A}_{\sigma,\tau}.$$

III) *For every probability measure Q absolutely continuous with respect to P, the minimal prenalty function satisfies the "cocycle condition"*

$$\alpha^{m}_{\nu,\tau}(Q) = \alpha^{m}_{\nu,\sigma}(Q) + E_Q(\alpha^{m}_{\sigma,\tau}(Q)|\mathcal{F}_{\nu}) \ \ Q \ a.s.$$

TIME CONSISTENCY

- condition on acceptance sets proved by Cheridito, Delbaen, Kupper (2006) They give also a characterization in terms of a concatenation condition.
- Characterization in terms of a "cocycle condition" on the minimal penalty function
  - Bion-Nadal (march 2006) in continuous time setting for dynamic risk measures continuous from below.
  - Föllmer and Penner (may 2006) in a discrete time setting for dynamic risk measures that admit a representation in terms of probability measures all equivalent to the reference probabilty
  - in the general case of dynamic risk measure continuous from above by Bion-Nadal (July 2006).

Key of the proof: extension of the theorem of representation

TIME CONSISTENCY

EXTENSION OF THEOREM OF REPRESENTATION

LEMMA

For every probability measure Q absolutely continuous with respect to P,

$$\rho_{\sigma,\tau}(X) = Q \; ess \; sup_{R \in \mathcal{M}^1_{\sigma,\tau}(Q)} (E_R(-X|\mathcal{F}_\sigma) - \alpha^m(R)) \;\; Q \; a.s. \qquad (4)$$

recall: $\mathcal{M}^1_{\sigma,\tau}(Q) = \{R \ll P \mid R_{|\mathcal{F}_\sigma} = Q \mid E_R(\alpha^m_{\sigma,\tau}(R)) < \infty\}$
remark: This lemma is not a consequence of the usual theorem of representation because if two random variables X and Y , $(\Omega, \mathcal{F}_\tau)$ measurable are equal Q almost surely, it is possible that the random variables $\rho_{\sigma,\tau}(X)$ and $\rho_{\sigma,\tau}(Y)$ are not equal Q a.s.

TIME CONSISTENT DYNAMIC RISK MEASURES FROM STABLE SETS

STABLE SET OF PROBABILTY MEASURES

DEFINITION

A set $\mathcal{Q}$ of probability measures all equivalent to P is stable if for every stopping times, $\nu \leq \sigma \leq \tau$, For every $Q \in \mathcal{Q}$, for every $R \in \mathcal{Q}$, there is $S \in \mathcal{Q}$ such that

$$\forall f \in L^\infty(\mathcal{F}_\tau),\ E_S(f|\mathcal{F}_\nu) = E_R(E_Q(f|\mathcal{F}_\sigma)|\mathcal{F}_\nu)\ P.a.s.$$

In this subsection a penalty function defined on a set $\mathcal{Q}$ of probability measures will always mean a family of maps $\alpha_{\sigma,\tau}$ defined on $\mathcal{Q}$ with values in $L^\infty(\Omega, \mathcal{F}_\sigma, P)$

CONDITIONS ON THE PENALTY FUNCTION

DEFINITION

A penalty function $\alpha_{\sigma,\tau}$

- is local if for every stopping times $\sigma \leq \tau$, for every A $\mathcal{F}_\sigma$-measurable, if $E_{Q_1}((X1_A|\mathcal{F}_\sigma) = E_{Q_2}(X1_A|\mathcal{F}_\sigma)\ P.a.s.\ \forall X \in L^\infty(\Omega, \mathcal{F}_\tau, P)$, then $1_A\alpha_{\sigma,\tau}(Q_1) = 1_A\alpha_{\sigma,\tau}(Q_2)\ P.a.s.$
- satisfies the cocycle condition if for every Q in $\mathcal{Q}$, for every stopping times $\sigma \leq \tau$,

$$\alpha_{\nu,\tau}(Q) = \alpha_{\nu,\sigma}(Q) + E_Q(\alpha_{\sigma,\tau}(Q)|\mathcal{F}_\nu)$$

SUFFICIENT CONDITION FOR TIME CONSISTENCY

The following theorem allows for the construction of many examples of time consistent dynamic risk measures:

THEOREM

Consider a family $\mathcal{Q}$ of probability measures on $(\Omega, \mathcal{F})$ all equivalent to P. Assume that $\mathcal{Q}$ is stable. Assume that the penalty function α defined on $\mathcal{Q}$ is local, satisfies the cocycle condition and is such that $\text{esssup}_{\mathrm{Q}\in\mathcal{Q}}(-\alpha_{\sigma,\tau}(\mathrm{Q}))$ *is essentially bounded. Then the dynamic risk measure $(\rho_{\sigma,\tau})_{0\leq\sigma\leq\tau}$ defined by*

$$\rho_{\sigma,\tau}(X) = \text{esssup}_{\mathrm{Q}\in\mathcal{Q}}\{\mathrm{E_Q}(-\mathrm{X}|\mathcal{F}_\sigma)) - \alpha_{\sigma,\tau}(\mathrm{Q})\}$$

is time-consistent.

This theorem was first proved by Delbaen (2003) in case of penalty identically equal to 0 on $\mathcal{Q}$

PLAN

EXAMPLE:ENTROPIC RISK

ENTROPIC DYNAMIC RISK MEASURE WITH THRESHOLDS

$(g_{s,t})_{0\leq s<t}$ stictly positive $\mathcal{F}_s$-measurable such that $\ln(g_{s,t})$ is bounded.

$$\begin{aligned}\rho_{s,t}(X) &= ess\ \inf\{Y \in \mathcal{E}_{\mathcal{F}_s} \;/\; E(e^{[-\alpha(X+Y)]}|\mathcal{F}_s) \leq g_{s,t}\} \\ &= \frac{1}{\alpha}[\ln E(e^{-\alpha X}|\mathcal{F}_s) - \ln(g_{s,t})].\end{aligned}$$

Minimal penalty: $\alpha^m_{s,t}(Q) = \frac{1}{\alpha}(E_P(\ln(\frac{dQ}{dP})\frac{dQ}{dP}|\mathcal{F}_s) - \ln(g_{s,t}))$.
$\rho_{s,t}$ is time consistent if and only if the functions $g_{s,t}$ are $\mathcal{F}_0$ measurable i.e. a.s. constant and satisfy the relation
$\forall\ r,s,t;\ 0 \leq r \leq s \leq t,\ \ \ln(g_{r,t}) = \ln(g_{r,s}) + \ln(g_{s,t})\ a.s.$
If $g_{s,t} = h(t-s)$ then $g_{s,t} = e^{\lambda(t-s)}$.
$\rho_{s,t}$ is normalized iff $g_{s,t} = 1\ \forall s \leq t$.
For the usual entropic dynamic risk measure: Barrieu and El Karoui (2005), Detlefsen and Scandolo (2005), Klöppel and Schweizer (2005)

EXAMPLE:BSDE

BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS $\mathcal{F}_t$ is the augmented filtration of a d dimensional Brownian motion. Assume that $g(t,z)$ is continuous and convex (in z), satisfies the condition of quadratic growth and $g(t,0)=0$. The associated BSDE,

$$\begin{aligned} -dY_t &= g(t,Z_t)dt - Z_t^* dB_t \\ Y_T &= X \end{aligned}$$

has a solution which gives rise to a dynamic risk measure $\rho_{s,T}(-X)=Y_s$.Barrieu and El Karoui have computed the minimal penalty associated to this dynamic risk measure using BMO continuous martingales.

STABLE SETS

We construct stable sets of probability measures from stable sets of martingales using the stochastic exponential $\mathcal{E}(M)$ of M. $\mathcal{E}(M)$ is the unique solution of $\mathcal{E}(M)_t = 1 + \int_0^t \mathcal{E}(M)_{s-} dM_s$

$$\mathcal{E}(M)_t = exp(M_t - \frac{1}{2}([M,M]^c)_t)\Pi_{s\leq t}(1 + \Delta M_s)e^{-\Delta M_s}$$

EXAMPLES USING CONTINUOUS MARTINGALES - Martingales with bounded quadratic variation

$$\mathcal{Q}_1 = \{Q_M \; ; \; \frac{dQ_M}{dP} = \mathcal{E}(M) \mid M \; continuous \; P \; martingale \; ;$$

$$[M,M]_\infty \in L^\infty(\Omega, \mathcal{F}, P)\}$$

is a stable set of probability measures all equivalent to P.

BMO MARTINGALES

BMO MARTINGALES

A square integrable cadlag martingale M is BMO if there is a constant c such that for every stopping time S,

$$E([M,M]_\infty - [M,M]_{S^-} | \mathcal{F}_S) \leq c^2\}$$

The smallest c is $||M||_{BMO}$

CASE OF CONTINUOUS BMO MARTINGALES

$$\mathcal{Q}_2 = \{Q_M \; ; \; \frac{dQ_M}{dP} = \mathcal{E}(M) \mid M \; continuous \; martingale \; ; ||M||_{BMO} < \infty\}$$

is stable.

STABLE SETS

STABLE SETS CONSTRUCTED FROM RIGHT CONTINUOUS BMO MARTINGALES

The construction of stable sets of probability measures using right continuous BMO martingales impose restrictions on the BMO norms.

LEMMA

Let $M^1, ..., M^j$ be a family of strongly orthogonal square integrable right continuous martingales. Let $(\Phi_i)_{1 \leq i \leq j}$ non negative predictable processes such that $\Phi_i M^i$ is a BMO martingale of BMO norm m^i. The set $\mathcal{M}$ of martingales of the form $\sum_{1 \leq i \leq j} H_i.N_i$ where H_i is a predictable process such that $|H_i| \leq \phi_i$ a.s. is a stable set of square integrable BMO martingales with norm BMO uniformly bounded by $(\sum_{1 \leq i \leq j} (m^i)^2)^{\frac{1}{2}} = m$. Denote $\mathcal{Q}_3$ the corresponding set of probability measures $(Q_M)_{M \in \mathcal{M}}$ of Radon Nikodym derivative $\frac{dQ_M}{dP} = \mathcal{E}(M)$.
If $m < \frac{1}{16}$, $\mathcal{Q}_3$ is a stable set of probability measures.

EXAMPLES FROM RIGHT CONTINUOUS MARTINGALES

PROPOSITION

Consider a stable set $\mathcal{Q} = \{Q_M \mid \frac{dQ_M}{dP} = \mathcal{E}(M) \;\; M \in \mathcal{M}\}$.(M BMO martingale) Let b a predictable process. For $0 \leq \sigma \leq \tau$ let

$$\alpha_{\sigma,\tau}(Q_M) = E_{Q_M}(\int_\sigma^\tau b_u d[M,M]_u)|\mathcal{F}_\sigma)$$

Assume that b is bounded and that the BMO norm of the elements of $\mathcal{M}$ are uniformly bounded by m. Assume furthermore that $m < \frac{1}{16}$ if the martingales in $\mathcal{M}$ are not all continuous. Then

$$\rho_{\sigma,\tau}(X) = \operatorname{esssup}_{\mathrm{Q_M} \in \mathcal{Q}(\mathcal{M})}(\mathrm{E_{Q_M}}(-\mathrm{X}|\mathcal{F}_\sigma) + \alpha_{\sigma,\tau}(\mathrm{Q_M})) \tag{5}$$

defines a time consistent dynamic risk measure.

This example generalizes the BSDE

PLAN

CADLAG MODIFICATION

Delbaen has proved that any coherent dynamic risk process such that $\alpha^m(P) = 0$ has a cadlag modification. We generalize this to normalized dynamic risk processes.

DEFINITION

A dynamic risk measure is non degenerate if $\forall A \in \mathcal{F}_\infty$, $\rho_{0,\infty}(\lambda 1_A) = \rho_{0,\infty}(0) \;\; \forall \lambda \in \boldsymbol{R}_+{}^*$ implies $P(A) = 0$.

LEMMA

Let $(\rho_{\sigma,\tau})_{0 \leq \sigma \leq \tau}$ a non degenerate normalized time consistent dynamic risk measure continuous from above. Let τ a stopping time. Every probability measure Q such that $\alpha^m_{0,\tau}(Q) = 0$ is equivalent to P on $(\Omega, \mathcal{F}_\tau)$.

CADLAG MODIFICATION

THEOREM

Consider a time consistent dynamic risk measure continuous from above, normalized and non degenerate. Assume that there is a probability measure with zero penalty ($\alpha_{0,\infty}(Q) = 0$). Every Q with zero penalty is equivalent to P and for every X there is a cadlag Q-supermartingale process Y such that for every stopping time $\sigma < \infty$, $\rho_{\sigma,\infty}(X) = Y_\sigma$ in $L^\infty(\Omega, \mathcal{F}_\sigma, P)$

Remark: such Q always exists if the dynamic risk measure is continuous from below.

Plan

FRAMEWORK FOR DYNAMIC PRICING

aim: Extend to a dynamic setting the approach based on observed prices in the market and non arbitrage. Construct a pricing procedure compatible with observed bid and ask prices for a reference family including instruments such as options for which only a bid price and an ask price are observed (at time 0)

REFERENCE FAMILY

- $d+1$ assets $(S^k)_{0\leq k\leq d+1}$ described by their locally bounded stochatic process on $(\Omega, \mathcal{F}_\infty, (\mathcal{F}_t))_{t\in R^+}, P)$. Among them S^0 that we take as numéraire. Thus $S^0_\tau = 1\ \forall \tau$.
- p assets $(Y^l)_{1\leq l\leq p}$ (such as options on one or several of the preceding assets) which are known only at one stopping time τ_l (maturity date). Y^l is modeled by one essentially bounded $\mathcal{F}_{\tau_l}$-measurable function.

Pricing Procedure

We want to assign to each financial position X defined at stopping time τ a dynamic bid price $\Pi_{\sigma,\tau}(X)$ and a dynamic ask price $-\Pi_{\sigma,\tau}(-X)$ (considering that selling X is the same as buying $-X$)
As the process describing the non risky asset is constant, this leads to the translation invariance property of $\Pi_{\sigma,\tau}$. The observation of the book of order leads to the conclusion that for λ large enough, $\Pi_{0,\tau}(\lambda X) < \lambda\Pi_{0,\tau}(X)$. Also taking into account the impact of diversification, it is natural to impose that $\Pi_{\sigma,\tau}$ is concave. Thus $-\Pi_{\sigma,\tau}$ is a normalized dynamic risk measure.
Moreover for $\nu \leq \sigma \leq \tau$ it is natural to ask that we get the same result for the bid price at time ν of a bounded $\mathcal{F}_\tau$ measurable financial instrument if we evaluate it either directly as $\Pi_{\nu,\tau}(X)$ or indirectly as $\Pi_{\nu,\sigma}(\Pi_{\sigma,\tau}(X)$. This is the time consistency condition.

PRICING PROCEDURE

DEFINITION

A dynamic pricing procedure $(\Pi_{\sigma,\tau})_{0\leq\sigma\leq\tau}$ on $(\Omega, \mathcal{F}_\infty, (\mathcal{F}_t)_{t\in R^+}, P)$ (where $\sigma \leq \tau$ are two stopping times) is the opposite of a normalized time consistent dynamic risk measure continuous from above. The bid (resp ask) price of $X \in L^\infty(\mathcal{F}_\tau)$ is modeled as $\Pi_{\sigma,\tau}(X)$ (resp $-\Pi_{\sigma,\tau}(-X)$)

STRONG ADMISSIBILITY

DEFINITION

A dynamic pricing procedure $(\Pi_{\sigma,\tau})_{0\leq\sigma\leq\tau}$ is strongly admissible with respect to the reference family $((S^k)_{0\leq k\leq d}, (Y^l)_{1\leq l\leq p})$ if

$$\forall 0 \leq k \leq d, \;\; \forall n \in \boldsymbol{Z} \;\; \forall 0 \leq \sigma \leq \tau$$

$$\text{if } \; S^k_\tau \in L^\infty(\mathcal{F}_\tau) \; \text{ then } \; \Pi_{\sigma,\tau}(nS^k_\tau) = nS^k_\sigma \;\; (A1)$$

$$\forall 1 \leq l \leq p \quad {C^l}_{bid} \leq \Pi_{0,\tau_l}(Y^l) \leq \; -\Pi_{0,\tau_l}(-Y^l) \leq {C^l}_{ask} \;\; (A2)$$

CHARACTERIZATION OF STRONG ADMISSIBILITY

Representation (recall)

$$\forall X \in L^\infty(\Omega, \mathcal{F}_\tau, P) \;\; \Pi_{0,\tau}(X) = inf_{Q \in \mathcal{M}_{0,\tau}} (E_Q(X) + \alpha^m_{0,\tau}(Q))$$

$$\mathcal{M}_{0,\tau} = \{Q \ll P \mid \alpha^m_{0,\tau}(Q) < \infty\}$$

CHARACTERIZATION OF STRONG ADMISSIBILITY

THEOREM

Let $\Pi_{\sigma,\tau}$ a pricing procedure. It is strongly admissible if and only if it satisfies the two following conditions

1. *Local martingale property: Let τ be a stopping time such that $S_\tau \in L^\infty(\mathcal{F}_\tau)$.*

$$\forall R \in \mathcal{M}_{0,\tau} \ \ (S^k_\sigma)_{\sigma \leq \tau} \ \ \textit{is a} \ \ R \ \textit{martingale}$$

   and also $\forall\ \sigma \leq \tau$,

$$\forall R \in \mathcal{M}^1_{\sigma,\tau}(P) \ \ E_R(S^k_\tau|\mathcal{F}_\sigma) = S^k_\sigma \ \ P\ a.s.$$

2. *for every $R \in \mathcal{M}_{0,\tau}$,*

$$\alpha^m_{0,\tau}(R) \geq \sup(0, \sup_{\{l \mid \tau_l \leq \tau\}} (C^{bid}_l - E_R(Y^l), E_R(Y^l) - C^{ask}_l))$$

CHARACTERIZATION OF STRONG ADMISSIBILITY

Denote $\mathcal{M}^0_{0,\tau} = \{Q \ll P \mid \alpha^m_{0,\tau}(Q) = 0\}$

COROLLARY

Consider a non degenerate pricing procedure continuous from below (or such that $\mathcal{M}^0_{0,\tau} \neq \emptyset$). For every $X \in L^\infty(\mathcal{F}_\tau)$, the process $\Pi_{\sigma,\tau}(X)$ has a cadlag modification. If the pricing procedure is admissible,

$$\forall Q_0 \in \mathcal{M}^0_{0,\tau} \;\; \forall 1 \leq l \leq p \;\; \textit{if} \;\; \tau_l \leq \tau \;\textit{then}\;\; C_l^{bid} \leq E_{Q_0}(Y^l) \leq C_l^{ask}$$

First fondamental theorem

This theorem is in the same vain as the Kreps Yan theorem as it is proved in Delbaen Schachermeyer (2006).

Cone of atteignable claims at zero cost

Definition

$$K = \{\sum_{i=1}^{n}\sum_{k=1}^{d}(h^k)_i(S^k_{\sigma_i}-S^k_{\sigma_{i-1}})+\sum_{l=1}^{p}(\gamma^l-\beta^l)Y^l+(\gamma^0-\beta^0)\ ;(h^k)_i \in L^\infty(\mathcal{F}_{\sigma_{i-1}}),$$

$$(\beta^l,\gamma^l)_{0\leq l\leq p} \in (\boldsymbol{R}^+)^{2(p+1)} |\ \sum_{l=1}^{p}(\beta^l C^l_{bid} - \gamma^l C^l_{ask}) + (\beta^0 - \gamma^0) \geq 0\}$$

Consider the set C of contingent claims dominated by an element of K.

$$C = K - L^\infty_+$$

NO FREE LUNCH

DEFINITION

The reference family $((S^k)_{1\leq k\leq d}, (Y^l)_{1\leq l\leq d})$ satisfies the no free lunch condition if the closure $\overline{C}$ of C with respect to the weak* topology of $L^\infty(\Omega,\mathcal{F},P)$ satisfies $\overline{C}\cap L^\infty_+(\Omega,\mathcal{F},P) = \{0\}$.

First fondamental theorem

Theorem

The following conditions are equivalent:

1. *The reference family* $((S^k)_{1\leq k\leq d}, (Y^l)_{1\leq l\leq d})$ *satisfies the no free lunch condition*
2. *There is an equivalent local martingale measure* Q *for the process* (S^k)*such that for every* $l \in 1, ..., p$, $C^l_{bid} \leq E_Q(Y^l) \leq C^l_{ask}$
3. *There is a non degenerate dynamic pricing procedure continuous from below, admissible with respect to the reference family* $((S^k)_{1\leq k\leq d}, (Y^l)_{1\leq l\leq d})$.

Example of admissible pricing procedure

Reference family reduced to S^k Assume that (S^k) satisfies the no free lunch condition. Let Q_0 an equivalent martingale measure for S^k. Assume that S^k is a square integrable martingale with respect to Q_0. Consider a stable set of right continuous Q_0-martingales strongly orthogonal to the martingale S^k , containing 0 of BMO norm uniformly bounded by $m < \frac{1}{16}$.
The set $\mathcal{Q}$ of probability measures of Radon Nikodym derivative $\frac{dQ_M}{dQ_0} = \mathcal{E}(M)$ is stable.
For every non negative predictable process b, the penalty

$$\alpha_{\sigma,\tau}(Q_M) = E_{Q_M}(\int_\sigma^\tau b_u d[M,M]_u)|\mathcal{F}_\sigma)$$

defines an admissible pricing procedure.

$$\Pi_{\sigma,\tau}(X) = ess\ inf_{Q_M \in \mathcal{Q}(\mathcal{M})}(E_{Q_M}(X|\mathcal{F}_\sigma) + \alpha_{\sigma,\tau}(Q_M))$$

Example of admissible pricing procedure

General reference family (S^k, Y^l)

Proposition

Assume that there is a probability measure Q_0 equivalent to P such that S is a square integrable Q_0-martingale and such that for every l,
$C^l_{bid} < E_{Q_0}(Y^l) < C^l_{ask}$.
There is a positive real number B such that every dynamic pricing procedure associated to a process $b \geq B$ as in the preceding example is strongly admissible.

PLAN

CONCLUSION

- Key property for dynamic risk measuring: Time consistency characterized by the cocycle condition of the minimal penalty function
- Any dynamic risk measure constructed from a stable set of probability measures and a local penalty satisfying the cocycle condition is time consistent. Using BMO martingales this gives rise to examples generalizing the BSDE
- For a normalized non degenerate dynamic risk measure, the dynamic risk process of any financial instrument has a cadlag modification.
- A pricing procedure associates to any financial instrument a bid (resp ask) pricing process. The bid pricing process is the opposite of a time consistent normalized dynamic risk measures .
- The admissibility of a pricing procedure with respect to a reference family (S_t^k, Y^l) implies that this pricing procedure has a dual representation in terms of equivalent martingale measures for the processes S_t^k
- A fundamental theorem of pricing is proved in that context