Microeconomic problems with law invariant concave utility functions

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Introduction

Goal: study standard microeconomic problems (demand, risk-sharing problems) when agents have concave quantile-based utility functions, differences with the standard expected utility framework (in insurance for instance).

involves optimization of such functionals (by means of convex analysis, differentiability properties etc...) to derive tractable computations and qualitative properties.

Plan of the talk

1. A class of utility functions
2. Demand problems
3. Efficient Risk-sharing
4. Equilibria

A class of utility functions

$(\Omega, \mathcal{F}, P)$ nonatomic probability space, $X \in L^\infty(\Omega, \mathcal{F}, P)$,

$$V(X) := \int_0^1 L(t, F_X^{-1}(t))dt + g(F_X^{-1}(0)) \tag{1}$$

with F_X^{-1} the quantile of X. Rank-Linear Utilities (RLU) (see Epstein-Chew, Green-Jullien).

Typical example: Choquet expectation with respect to a convex distortion $f : [0, 1] \to [0, 1]$ continuous or discontinuous at 1.
Continuous case:

$$E_f(X) = \int_0^1 f'(1-t) F_X^{-1}(t) dt \tag{2}$$

obtained from (1) by setting $g = 0$ and $L(t, x) = f'(1-t)u(x)$.
Yaari utility functions or (up to minus sign) comonotonic law invariant risk measures (e.g. expected shortfall) see Barrieu-ElKaroui, Schied, Jouini-Schachermayer-Touzi.

Rank dependendent expected utility (RDU):

$$E_f(u(X)) = \int_0^1 f'(1-t) u(F_X^{-1}(t)) dt$$

Discontinuous distortion

$$E_f(u(X)) = (1 - f(1^-))u(F_X^{-1}(0)) + \int_0^1 f'(1-t)u(F_X^{-1}(t))dt \quad (3)$$

defines a utility of the type (1) with $L(t,x) = f'(1-t)u(x)$ and $g(x) = (1 - f(1^-))u(x)$ and $f(1^-)) = \lim_{x \uparrow 1} f(x)$.

Interpretation: aversion to worst case.

Stochastic dominance

X dominates Y in the sense of second order stochastic dominance (SSD), if $E(u(X)) \geq E(u(Y))$, for every utility function $u : \mathbb{R} \to \mathbb{R}$ concave nondecreasing.

X strictly dominates Y in the sense of SSD if $E(u(X)) > E(u(Y))$, for every utility function $u : \mathbb{R} \to \mathbb{R}$ strictly concave nondecreasing.

A utility function V (strictly) preserves SSD if $V(X) \geq V(Y)$ $(V(X) > V(Y))$ whenever X (strictly dominates) Y in the sense of SSD. V monotone if $V(X) \geq V(Y)$ whenever $X \geq Y$.

$$V_L(X) := \int_0^1 L(t, F_X^{-1}(t))dt \tag{4}$$

Proposition 1 *Let* $(\Omega, \mathcal{F}, P)$ *be non atomic and* V_L *be of type (4). Let* $L \in C^2([0,1] \times \mathbb{R})$*. The following are equivalent:*

1. V_L *is SSD preserving,*
2. $\partial_x L \geq 0$, $\partial_{xx} L \leq 0$ *and* $\partial_{tx} L \leq 0$ *on* $[0,1] \times \mathbb{R}$
3. V_L *is concave, monotone and* $\sigma(L^\infty(\Omega), L^1(\Omega))$ *upper semi-continuous.*

From now on, we assume that L satisfies the previous properties.

Strict SSD holds if $L(t, .)$ is strictly concave or $\partial_{tx} L < 0$.

Comonotonicity

Key notion of comonotonicity

X and Y are comonotone whenever

$$(X(\omega) - X(\omega'))(Y(\omega) - Y(\omega)) \geq 0, \; P \otimes P \text{ a.s.}$$

(X and Y anticomonotone: X and $-Y$ comonotone).

also means that X and Y are nondecreasing 1-Lipschitz functions of their sum.

Under the assumptions of Proposition 1, one has:

$$V_L(X) = \inf_{U \text{ uniform}} E(L(U, X)).$$

and

$$V_L(X) = E(L(U, X))$$

for every U uniformly distributed and comonotone with X.

Differentiability

Assume that L is of class C^2 and :

$$\partial_x L \geq 0,\ \partial_{xx} L \leq 0,\ \partial_{tx} L < 0 \text{ on } [0,1] \times \mathbb{R}.$$

The superdifferential of V_L at $X \in L^\infty(\Omega)$ denoted $\partial V_L(X)$ is defined by:

$$\partial V_L(X) := \{\mu \in (L^\infty)^{'} \colon V_L(Y) - V_L(X) \leq \langle \mu, Y - X \rangle\,,\ \forall\, Y \in L^\infty\}$$

V_L is Gâteaux-differentiable at X if:

$$Y \in L^\infty(\Omega) \mapsto D^+ V_L(X;Y) := \lim_{t \to 0^+} \frac{1}{t}[V_L(X + tY) - V_L(X)]$$

defines a continuous linear form on $L^\infty(\Omega)$, denoted $V_L'(X)$.

Theorem 1 *Let $X \in L^\infty(\Omega)$, then the following holds:*

1. $\partial V_L(X) := \overline{\text{co}}\{\partial_x L(U, X),\ U$ *uniform, comonotone with* $X\}$ *where* $\overline{\text{co}}$ *denotes closed convex hull operation for the* $L^1(\Omega)$ *topology,*

2. *any element of* $\partial V_L(X)$ *is anticomonotone with* X*,*

3. *defining* $\Omega_X := \{\omega \in \Omega \ : \ F_X$ *is continuous at* $X(\omega)\}$*, for every* $\Psi \in \partial V_L(X)$ *and a.e.* $\omega \in \Omega_X$*, one has* $\Psi(\omega) = \partial_x L(F_X(X(\omega)), X(\omega))$*,*

4. V_L *is Gâteaux-differentiable at* X *if and only if* F_X *is continuous, in which case, one has:*

$$\partial V_L(X) = \{V_L'(X)\} = \{\partial_x L(F_X \circ X, X)\}.$$

Demand Problems

$L_+^\infty(\Omega, \mathcal{F}, P)$: set of bounded state contingent consumptions. Let $\psi \in L_+^1(\Omega)$ with $E(\psi) = 1$ be a pricing density. Consider an agent with utility $V : L^\infty(\Omega) \to \mathbb{R} \cup \{-\infty\}$ and income $w > 0$. Agent's demand for state contingent claims is the solution to :

$$(\mathcal{D}) \ \ \sup\{V(X) : E(\psi X) \leq w, \, X \geq 0\}. \tag{5}$$

Assume V is strictly SSD preserving (hence law invariant) and u.s.c. concave and that the cumulative F_ψ is continuous (or equivalently F_ψ^{-1} is strictly increasing).

Quantile reformulation

Intuition suggests that the demand problem may be restricted to the class of nonincreasing function of the price i.e. (5) may be restricted to claims of the form $X = x(1 - F_\psi(\psi))$ with $x \in \mathcal{A}$ and

$$\mathcal{A} := \{x : [0,1] \to \mathbb{R}_+, \; x \text{ nondecreasing}\}.$$

note that $X = x(1 - F_\psi(\psi)) \Rightarrow x = F_X^{-1}$.

Indeed, setting $q(t) := F_\psi^{-1}(1-t)$ and $v(x) = V(x \circ U)$ (U uniformly distributed), one has:

Proposition 2 *$\bar{X}$ is a solution of $(\mathcal{D})$ iff $\bar{X} = \bar{x}(1 - F_\psi(\psi))$ and $\bar{x}$ is a solution of :*

$$(\tilde{\mathcal{D}}) \ \sup\{v(x) : x \in \mathcal{A},\ x \textit{ bounded and } \int_0^1 q(t)x(t)dt \leq w\}$$

Equivalently, there exists $\lambda > 0$ such that $\overline{x} = \overline{x}_\lambda$, solution of

$$\sup_{y \in \mathcal{A}} v(y) - \lambda \int_0^1 q(t)y(t)dt \tag{6}$$

and $\int_0^1 q(t)\overline{x}_\lambda(t)dt = w$.

If we further specify V of the form (1), we have to solve for fixed $\lambda > 0$:

$$(\mathcal{P}_\lambda)\ \sup_{x\in\mathcal{A}} \int_0^1 (L(t, x(t)) - \lambda q(t)x(t))dt + g(x(0))$$

and then find λ such that the budget constraint holds.

Monotonicity constraint: a classical problem in the adverse selection literature (Mussa-Rosen, Guesnerie-Laffont, Rochet).

Assumptions

- $L \in C^1([0,1] \times \mathbb{R}_+^*, \mathbb{R})$, $g \in C^1(\mathbb{R}_+^*, \mathbb{R})$,
- for every $t \in [0,1]$, $L(t,.)$ is strictly concave increasing on $\mathbb{R}_+^*$, g is strictly concave increasing on $\mathbb{R}_+^*$, and:

$$\lim_{x \to +\infty} \sup_{t \in [0,1]} \frac{\max(L(t,x),0)}{x} = 0 \tag{7}$$

- there exists $q_0 > 0$ such that $q \geq q_0$ on $[0,1]$, q is continuous on $[0,1]$,
- the following function:

$$\widetilde{x}_\lambda(t) := \text{argmax}_{x \in \mathbb{R}_+}(L(t,x) - \lambda q(t)x).$$

  $\widetilde{x}_\lambda$ is Lipschitz continuous on $[0,1]$,

- **either**:

$$\lim_{x \to 0^+} g'(x) = +\infty, \ \textbf{or} \tag{8}$$

$$\lim_{(t,x)\to(0^+,0^+)} \partial_x L(t,x) = \lim_{\varepsilon \to 0^+} \int_0^\delta \partial_x L(t,\varepsilon)dt = +\infty, \ \forall \delta \in (0,1). \tag{9}$$

Under the previous assumptions, the demand problem admits a unique solution $\overline{x}$, and $(\mathcal{P}_\lambda)$ admits a unique solution $\overline{x}_\lambda$ for every $\lambda > 0$, moreover $\overline{x}_\lambda$ is Lipschitz continuous, hence differentiable a.e., for every $\lambda > 0$, and $\overline{x}_\lambda(0) > 0$.

Optimality conditions

For $\lambda > 0$, $\overline{x}_\lambda$, solution of $(\mathcal{P}_\lambda)$ is characterized by

Proposition 3 *Let $\lambda > 0$ and $x \in \mathcal{A} \cap L^\infty$ and let Λ be defined for every $t \in [0,1]$ by:*

$$\dot{\Lambda}(t) := \partial_x L(t, x(t)) - \lambda q(t) \text{ and } \Lambda(1) = 0 \tag{10}$$

then $x = \overline{x}_\lambda$ if and only if x is differentiable a.e. and:

$$\begin{array}{ll} (i) & \Lambda \geq 0, \text{ and } \Lambda(t)\dot{x}(t) = 0 \text{ a.e.}, \\ (ii) & x(0) > 0 \text{ and } \Lambda(0) = g'(x(0)). \end{array}$$

Allows to solve (very simply in some cases e.g. RDU) $(\mathcal{P}_\lambda)$ by the so-called *ironing* procedure.

Examples

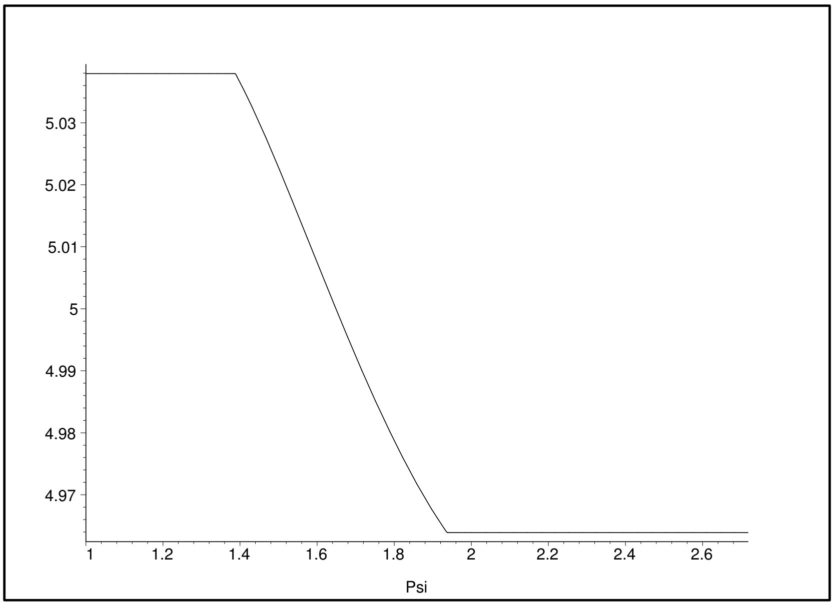
Example 1

A continuous RDU example

$u(x) = \ln(x)$, $F_\psi^{-1}(t) = q(1-t) = e^t$ and that the distortion f is given by:

$$f(t) = \frac{9}{2} + e^t(-\frac{9}{2} + \frac{19}{2}t - \frac{9}{2}t^2 + t^3)$$

$\Rightarrow$ two flat zones (strangled demand)

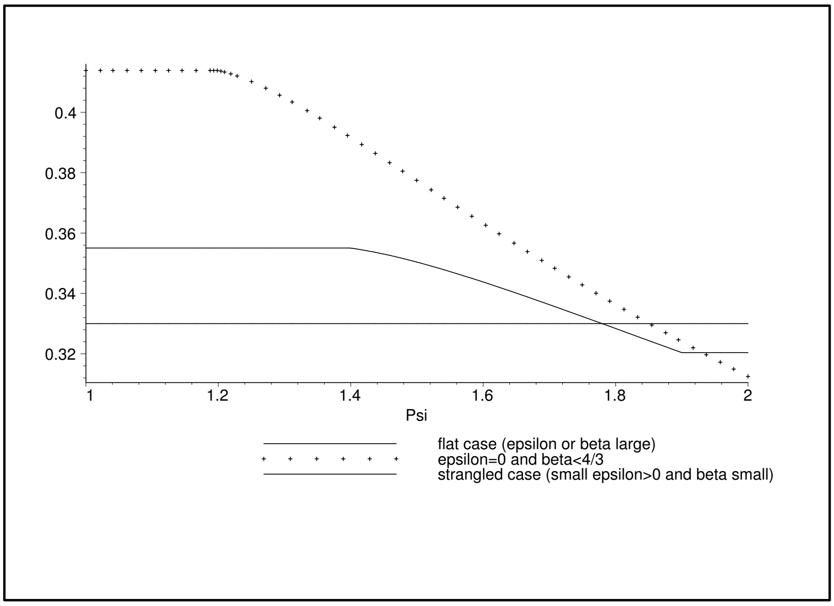

5.03
5.02
5.01
5
4.99
4.98
4.97
1
1.2
1.4
1.6
1.8
2
2.2
2.4
2.6
Psi

Example 2

RDU with discontinuous distortions,
$L(t,x) = (1-\varepsilon)\beta(1-t)^{\beta-1}\ln(x)$, $g(x) = \varepsilon\ln(x)$, $\beta \geq 1$, $\varepsilon \in [0,1)$. Prices uniformly distributed on $[1,2]$, i.e. $q(t) = 2-t$

If $\varepsilon = 0$: flat demand for $\beta \geq 4/3$, flat demand only for low prices otherwise.

If $\varepsilon > 0$: demand always flat for high prices, totally flat for $\beta \geq \beta(\varepsilon)$ ($\beta(.)$ nonincreasing).



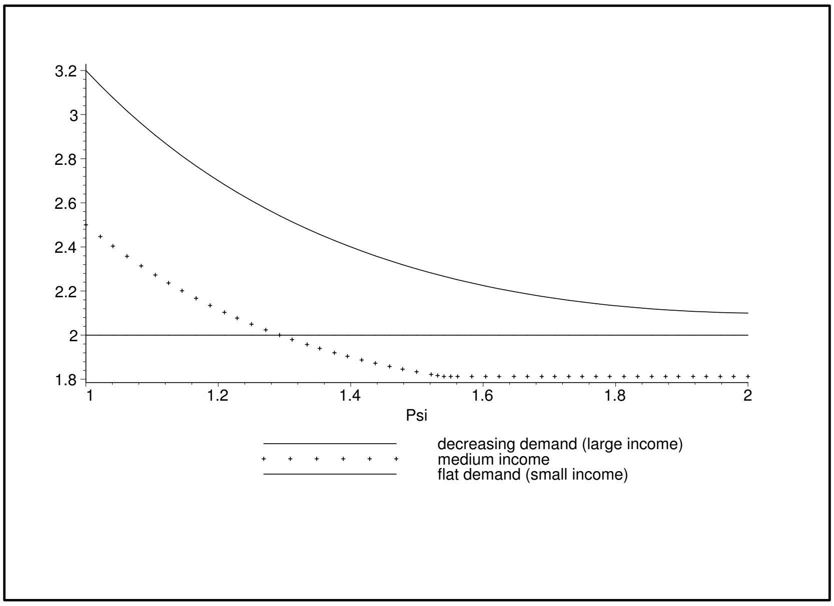
0.4
0.38
0.36
0.34
0.32
1
1.2
1.4
1.6
1.8
2
Psi
flat case (epsilon or beta large)
epsilon=0 and beta<4/3
strangled case (small epsilon>0 and beta small)

Example 3

RLU: $L(t,x) = \ln(t+x)$ and as previously, prices are uniformly distributed on $[1,2]$, i.e. $q(t) = 2 - t$.

- flat demand when $w \leq 3/(2e^{3/2} - 2)$,
- decreasing then constant for high priceswhen $w \in (3/(2e^{3/2} - 2), 10/3]$,
- decreasing, when $w > 10/3$.

RLU: allows richer income effects than RDU (in the lattercase, the number of flat zones is independent of w).


3.2
3
2.8
2.6
2.4
2.2
2
1.8
1
1.2
1.4
1.6
1.8
2
Psi
decreasing demand (large income)
medium income
flat demand (small income)

Efficient risk-sharing

Two agents framework utilities V_1, V_2, aggregate endowment $X_0 \geq 0$, weight $\lambda \in (0,1)$

$$\sup\{\lambda V_1(X) + (1-\lambda)V_2(X_0 - X) \ : \ 0 \leq X \leq X_0\}$$

Assume that F_{X_0} is continuous and V_1, V_2 are strictly SSD, concave u.s.c.

Quantile reformulation

Key observation: noncomonotone pairs are dominated by comonotone ones. One may therefore restrict attention to wealths of the form $(x(F_{X_0}(X_0)),(x_0-x)(F_{X_0}(X_0)))$ where $x_0 := F_{X_0}^{-1}$ and $x \in \mathcal{A}_2$:

$\mathcal{A}_2 := \{x : [0,1] \to \mathbb{R} \ : \ 0 \leq x \leq x_0, \ x$ and $x_0 - x$ nondecreasing$\}$.

If V_1 and V_2 are RLU, the Pareto Problem amounts to:

$$\sup_{x \in \mathcal{A}_2} \int_0^1 L_\lambda(t, x(t))dt + g_\lambda(x(0)). \tag{11}$$

where:

$$L_\lambda(t,x) = \lambda L_1(t,x) + (1-\lambda)L_2(t, x_0(t) - x)$$

and:

$$g_\lambda(x) = \lambda g_1(x) + (1-\lambda) g_2(x_0(0) - x).$$

Optimality conditions

Under assumptions similar to those of the demand problem, the solution of the previous problem is characterized by:

Proposition 4 *Let $x \in \mathcal{A}_2$ and Λ_λ be defined by:*

$$\dot{\Lambda}_\lambda(t) = \partial_x L_\lambda(t, x(t)), \text{ for all } t \in [0,1], \Lambda_\lambda(0) = 0.$$

Then $x = \overline{x}_\lambda$, the solution of (11), iff the following conditions hold:

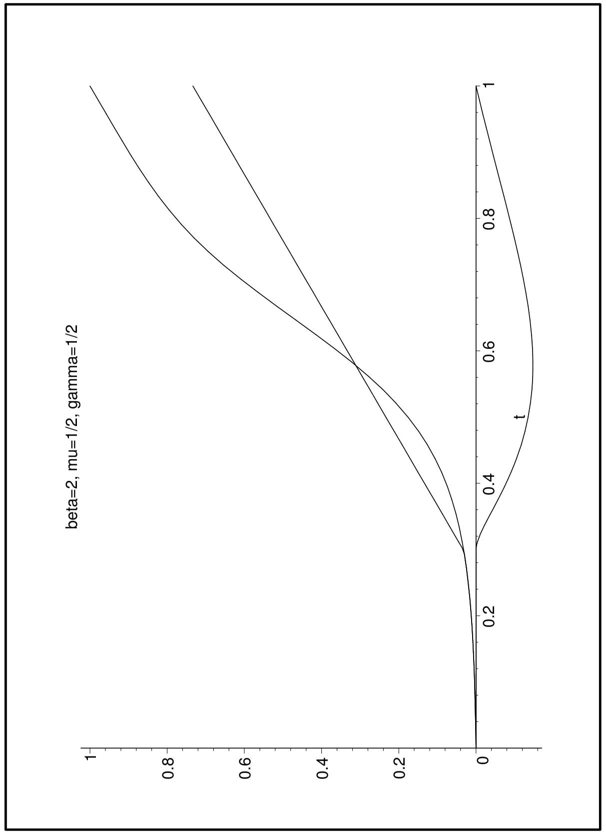
(i) $(\Lambda_\lambda - \Lambda_\lambda(1))_+ \dot{x} = 0$, *a.e.*

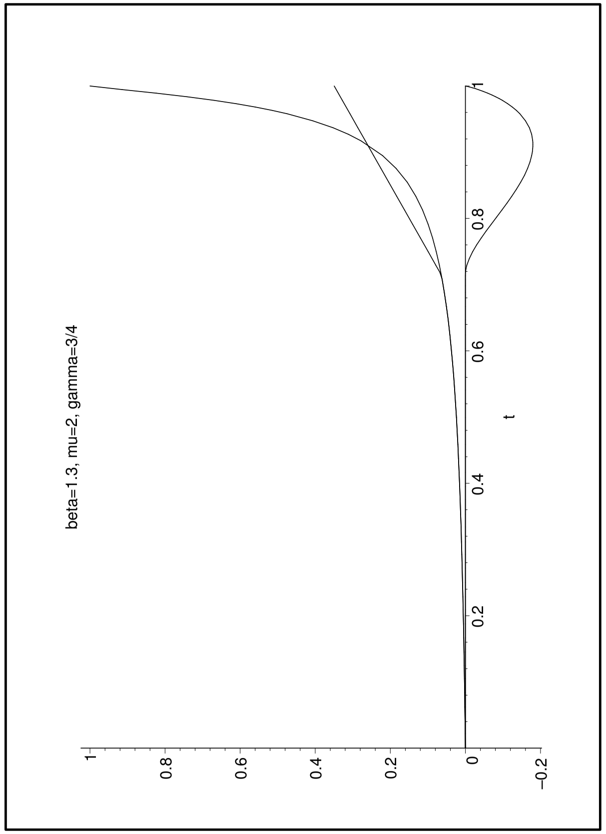
(ii) $(\Lambda_\lambda - \Lambda_\lambda(1))_- (\dot{x}_0 - \dot{x}) = 0$, *a.e.*

(iii) $0 < x(t) < x_0(t)$ *for all* $t \in [0,1]$, *and* $\Lambda_\lambda(1) = -g'_\lambda(x(0))$.

Example

Risk sharing problem between an expected utility maximizer agent and an RDU agent. Further assume that the aggregate endowment X_0 is uniformly distributed on $[0, 1]$. Specification $L_1(x) = x^\gamma$, $L_2(t, x) = \beta(1 - t)^{\beta - 1}x^\gamma$, $\gamma \in (0, 1)$, $\beta \geq 1$, $g_1 = g_2 = 0$.


beta=2, mu=1/2, gamma=1/2
t
0
0.2
0.4
0.6
0.8
1


beta=1.3, mu=2, gamma=3/4
t
1
0.8
0.6
0.4
0.2
0
−0.2
0.2
0.4
0.6
0.8
1

Equilibria

Aggregate endowment X_0, (X_1, X_2) feasible iff $X_1 + X_2 = X_0$ and $0 \leq X_1 \leq X_0$.

A triple $(X_1^*, X_2^*, \Psi^*) \in (L_+^\infty)^2 \times (L^\infty)'_+$ with (X_1^*, X_2^*) feasible is an equilibrium with transfer payments if for $i = 1, 2$, X_i^* solves

$$\max V_i(X_i) \text{ s.t. } \langle \Psi^*, X_i \rangle \leq \langle \Psi^*, X_i^* \rangle \,, \; X_i \in L_+^\infty$$

$(X_1^*, X_2^*, \Psi^*) \in (L_+^\infty)^2 \times (L^\infty)'_+$ with (X_1^*, X_2^*) feasible is an equilibrium if it is an equilibrium with transfer payments such that

$$\langle \Psi^*, X_1^* \rangle = \langle \Psi^*, W_1 \rangle \,. \tag{12}$$

where W_1, W_2 are the agents'initial endowments $(X_0 = W_1 + W_2)$.

If utilities are superdifferentiable, (X_1^*, X_2^*, Ψ^*) (with (X_1^*, X_2^*) feasible) is an interior equilibrium with transfer payments iff there exists $\lambda \in (0,1)$ and $\alpha > 0$ such that

$$\lambda \partial V_1(X_1^*) \cap (1-\lambda)\partial V_2(X_2^*) \neq \emptyset \text{ and}$$

$$\alpha \Psi^* \in \lambda \partial V_1(X_1^*) \cap (1-\lambda)\partial V_2(X_2^*).$$

In particular, (X_1^*, X_2^*) solves the problem

$$\sup\{\lambda V_1(X_1) + (1-\lambda)V_2(X_2) \ : \ (X_1, X_2) \text{ feasible}\}$$

hence is Pareto efficient.

Parameterization

RLU case

Proposition 5 *Assume that* $g_1 = g_2 = 0$, *then* $(X^*, X_0 - X^*, \Psi^*)$ *is an interior equilibrium with transfer payments iff there exists* $\lambda \in (0,1)$ *such that*

1. $X^* = \overline{x}_\lambda(F_{X_0}(X_0))$,
2. $\Psi^* \in L^1_+$ *and* Ψ^* *is proportional to* $p_\lambda(F_{X_0}(X_0))$ *with*

$$p_\lambda(t) = \begin{cases} \lambda \partial_x L_1(t, \overline{x}_\lambda(t)) \text{ on the support of } d\overline{x}_\lambda \\ (1-\lambda)\partial_x L_2(t, (x_0 - \overline{x}_\lambda)(t)) \text{ on its complement} \end{cases}$$