
Risk Measures and Optimal Design Under Interest Rate Ambiguity

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Agenda

- ★ Main motivation of this study and some references.
- ★ **Static Framework :**
 - ◇ Basic recalls and new developments on monetary risk measures.
 - ◇ Cash invariance and discounting : cash and forward risk measures
- ★ **Cash sub-linear risk measures :**
 - ◇ Definition and dual representation
 - ◇ Extensions into cash-invariant risk measure on defaultable risks.
- ★ **Optimal Derivatives design with time to default uncertainty**
 - ◇ Inf-convolution, and optimal structure.
- ★ **Dynamic framework via BSDE's : Properties of the coefficient**
 - ◇ Inf-convolution of the coefficients and optimal structure.

Introduction and Motivation

Risk management

- New developments, new regulatory rules.
- Credit risk regulation : Bale II

Question : How to take into account ambiguity on interest rates, or time to default ?. Classical **monetary risk measures** are not adapted tools to do that.

Design of new financial products in this context

- How the design financial contracts to hedge non-financial risk :
"cat-bonds", weather derivatives...
- *Illiquid* instruments, with an underlying asset which is *not traded* on financial markets.
- Credit derivatives

How to separate the specific risk of the discount (defaultable) risk

Some related works (among many others...!!)

★ *Insurance literature on optimal policy design* : Borch (1962), Bühlmann (1970), Raviv (1979), Gerber (1980)...

★ *Indifference pricing* (often in an exponential framework) : Hodges-Neuberger(1989), Davis(1997), Rouge-El Karoui(2000), Becherer (2001), Delbaen et al (2002), Musiela-Zariphopoulou (2004)....

★ *Risk measures*

⇒ (seminal papers) : Deprez-Gerber(1985), Artzner et al.(1999), Delbaen(2000), Carr-Géman-Madan(2001), Föllmer-Schied(2002), Frittelli-Gianin(2002)...

⇒ (multi-period setting) : Scandolo (2003), Artzner et al.(2004), Cheridito et al.(2004), Riedel (2004), Weber(2005), Detlefsen-Scandolo (2005), Frittelli-Scandolo (2005), Klöppel-Schweizer (2005)...

★ *BSDEs and finance* : El Karoui-Quenez(1996), El Karoui-Peng-Quenez(1997), Peng(1997, 2003), Mania-Schweizer(2004)...

Convex Risk measures : Basic properties

Definition : Let (Ω, \mathcal{F}) be a standard measurable space and \mathcal{X} the linear space of bounded functions (including constant functions).

The functional ρ is a monetary risk measure if it satisfies :

- Convexity, and Decreasing monotonicity ;
- Translation invariance : $\forall X \in \mathcal{X}, \forall m \in \mathbb{R}, \quad \rho(X + m) = \rho(X) - m.$

In particular, $\rho(X + \rho(X)) = 0.$

Comments

- $\Rightarrow \rho(X)$ can be also interpreted as the *minimal capital requirement* to be added to the position X to make it "acceptable" : the new position $X + \rho(X)$ does not carry any risk with positive measure.
- $\Rightarrow -\rho(X)$ can be seen as the maximum amount the agent is ready to pay to hold the risk X in her portfolio.
- $\Rightarrow -\rho(X)$ can be seen as the *indifference buyer's price*.

Dual representation

The convexity of the framework leads to an "explicit" representation

Theorem : *There exists a penalty function α taking values in $\mathbb{R} \cup \{+\infty\}$ such that :*

$$\begin{aligned} \forall \Psi \in \mathcal{X}, \quad \rho(\Psi) &= \sup_{\mathbf{Q} \in \mathcal{M}_{1,f}} \{ \mathbb{E}_{\mathbf{Q}}[-\Psi] - \alpha(\mathbf{Q}) \} \\ \forall \mathbf{Q} \in \mathcal{M}_{1,f}, \quad \alpha(\mathbf{Q}) &= \sup_{\Psi \in \mathcal{X}} \{ \mathbb{E}_{\mathbf{Q}}[-\Psi] - \rho(\Psi) \} \end{aligned}$$

where $\mathcal{M}_{1,f}$ is the set of all additive measures on (Ω, \mathcal{F}) .

Moreover, the supremum is attained in the first equation in $\mathcal{M}_{1,f}$ **Example :**

The entropic risk measure is defined as $e_{\gamma}(\Psi) = \gamma \ln \mathbb{E}_{\mathbb{P}} \left[\exp \left(-\frac{1}{\gamma} \Psi \right) \right]$,

where γ is the *risk tolerance coefficient*.

Its dual representation is :

$$e_{\gamma}(\Psi) = \sup_{\mathbf{Q} \in \mathcal{M}_1} (\mathbb{E}_{\mathbf{Q}}[-\Psi] - \gamma h(\mathbf{Q}|\mathbb{P}))$$

where \mathcal{M}_1 is the set of all probability measures on (Ω, \mathcal{F}) and $h(\mathbf{Q}|\mathbb{P})$ is the relative entropy of \mathbf{Q} w.r. to \mathbb{P} .

Cash-Invariance and Discounting

The definition of monetary risk measure implicitly assumes that future risk position and risk measure are expressed in the **same numéraire**.

Convention : $1_T = 1$ unit of cash available at time T .

$D_{0,T}$ is the random discount factor.

Spot risk measure (Foellmer-Schied,2004) et Cash-invariance

$$\rho_0(\mathbf{D}_{0,T}\mathbf{X}_T + \mathbf{m}\mathbf{1}_0) = \rho_0(D_{0,T}X_T) + \rho_0(m1_0), \quad \rho_0(m1_0) = m1_0$$

Forward risk measure (Rouge-El Karoui(2000))

$$\rho_T(X_T) = \rho_T(X_T + m1_T) = \rho_T(X_T) + \rho_T(m1_T) \quad \text{and} \quad \rho_T(1_T) = -1$$

Spot risk measure and Forward risk measure

Suppose **a zero-coupon bond** with T maturity is traded $B(0, T)$ on the market.

Then $\mathcal{R}_0^{\rho_T}(\mathbf{D}_{0,T}\mathbf{X}_T) := \mathbf{B}_{0,T}\rho_T(\mathbf{X}_T)$ is a cash invariante risk measure iff ρ_T satisfies the calibration constraint :

$$\rho_T(\mathbf{D}_{0,T}^{-1}) = -\mathbf{B}(0, T).$$

Then, for any \mathbb{Q}_T in the domain of α_T ,

$$\mathbb{E}_{\mathbb{Q}_T}(D_{0,T}) = B(0, T)$$

\mathbb{Q}_0 belongs to $\text{dom}(\alpha_0)$ iff \mathbb{Q}_T defined by

$$\frac{d\mathbb{Q}_T}{d\mathbb{Q}_0} = D_{0,T}/B(0, T)$$

belongs to $\text{dom}(\alpha_T)$

Cash sub-additive Risk measure

Observe that, if $\mathcal{R}(X_T) = \rho(X_T D_{0,T})$, then

$$\begin{aligned} \forall m \geq 0, \forall X_T \in \mathcal{X}_T, \quad & \mathcal{R}(X_T + m1_T) = \rho(X_T D_{0,T} + mD_{0,T}) \\ & \geq \rho(X_T D_{0,T} + m1_0) = \rho(X_T D_{0,T}) - m1_0 = \mathcal{R}(X_T) - m1_0 \end{aligned}$$

In other words, for any $X_T \in \mathcal{X}_T$ the function $m \in \mathbb{R} \mapsto \mathcal{R}(X_T + m1_T) + m1_0$ is nondecreasing. \mathcal{R} is said to be **cash sub-additive**.

Def : Any convex, decreasing, cash sub-additive functional defined on \mathcal{X}_T is called *Cash sub-additive Risk measure*.

Monetary risk measure extension

The key property is that $\hat{\mathcal{R}}(X_T, x) := \mathcal{R}(X_T - x1_T) - x1_0$ is **cash invariante** as a functional of (X_T, x) , since

$$\hat{\mathcal{R}}(X_T + m, x + m) = \mathcal{R}((X_T + m - (x + m)1_T) - (x + m)1_0) = \hat{\mathcal{R}}(X_T, x) - m1_0$$

Example : Cash sub-additive r.m. under ambiguous discount factors

Hyp :

- unknown D_T ranging between two constant : $0 \leq d_L \leq d_H \leq 1$
 - regulator endowed with $\bar{\rho}$, a spot-cash invariant risk measure on
- Regulator assesses risk of future payoff X_T in the **worst case interest rates scenario** :

$$\mathcal{R}^{\bar{\rho}, D}(X_T) := \sup_{D_T \in \{0 \leq d_L \leq D_T \leq d_H \leq 1\}} \{\bar{\rho}(D_T X_T)\}.$$

$\mathcal{R}^{\bar{\rho}, D}$ is a cash sub-additive risk measure

- convex and monotone decreasing as supremum of risk measures
- cash sub-additive :

$$\mathcal{R}^{\mathbf{D}, \bar{\rho}}(\mathbf{X}_T) = \bar{\rho}(-\mathbf{v}(\mathbf{X}_T)), \quad \mathbf{v}(\mathbf{x}) = -(\mathbf{d}_L \mathbf{x}^+ - \mathbf{d}_H (-\mathbf{x})^+)$$

Proposition

Let v be a real, convex, decreasing function such : $v'_x \in [-1, 0]$ and $v(0) = 0$,
and $\bar{\rho}$ cash invariant r.m. on \mathcal{X}_T with minimal penalty function $\alpha^{\bar{\rho}}$.

Then $\mathcal{R}^{\bar{\rho},v}(X_T) := \bar{\rho}(-v(X_T))$ is a cash sub-additive r.m.

$v'_x < 0$ and convexity of v are used to provide the convexity and decreasing
monotonicity of $\bar{\rho}(-v)$.

$v'_x > -1$ is used to provide the cash sub-additivity

Moreover, as v is a continuous function,

$$\mathcal{R}^{\bar{\rho},v}(X_T) = \sup_{D_T \in [0,1]} \{ \bar{\rho}(D_T X_T + \beta(-D_T)) \mid 0 \leq D_T \leq 1 \}$$

$\beta(y) = \sup_{x \in \mathbb{R}} \{xy - v(x)\}$, convex Fenchel transform of v , and

$$\mathcal{R}^{\bar{\rho},v}(\mathbf{X}_T) = \sup_{\bar{Q} \in \mathcal{Q}_{1,f}, D_T \in [0,1]} \left\{ \mathbb{E}_{\bar{Q}}[-D_T X_T] - \alpha^{\bar{\rho},v}(\bar{Q}, D_T) \mid \mathbf{0} \leq \mathbf{D}_T \leq \mathbf{1} \right\}$$

where $\alpha^{\bar{\rho},v}(\bar{Q}, D_T) := \alpha^{\bar{\rho}}(\bar{Q}) + \mathbb{E}_{\bar{Q}}[\beta(-D_T)]$

(2)

Remark : The dependence of $\alpha^{\bar{\rho}, v}(\bar{Q}, D_T) = \alpha^{\bar{\rho}}(\bar{Q}) + \mathbb{E}_{\bar{Q}}[\beta(-D_T)]$ on D_T and \bar{Q} is **mixed**

Example : cash sub-additive r.m. under ambiguous discount factor

$$\mathcal{R}^{\bar{\rho}}(X_T) = \sup_{D_T \in \bar{Q} \in_{1,f}} \left\{ \mathbb{E}_{\bar{Q}}[-D_T X_T] - (\alpha^{\bar{\rho}}(\bar{Q}) + l(D_T)) \right\}$$

l convex indicator function of $\{D_T \in : 0 \leq d_L \leq D_T \leq d_H \leq 1\}$

Example : $\bar{\rho}(X_T) = \rho_{max}(X_T) = \sup_{Q \in_{1,f}} \mathbb{E}_Q[-X_T]$ worst case r.m.

$$\mathcal{R}^{\bar{\rho}, v} = \rho_{max}(-v(X_T)) = \| -v(X_T) \|_{\infty}$$

$$\mathcal{R}^{\bar{\rho}, v} = \rho_{max}(-v(X_T)) = \sup_{D_T \in \bar{Q} \in_{1,f}} \left\{ \mathbb{E}_{\bar{Q}}[-D_T X_T] - \mathbb{E}_{\bar{Q}}[\beta(-D_T)] \right\}$$

Cash sub-additive risk measure suitable to assess financial or insurance risks not only in the framework of risky interest rates.

Example : Put Premium Risk Measure

Let $r \geq 1$ be the riskfree interest rate to transfer money from T to 0.

The premium of a put option with strike price K and maturity T

$$\mathcal{R}(\mathbf{X}_T) = \frac{1}{r} \mathbb{E}[(K - \mathbf{X}_T)^+]$$

is a **cash-sub-additive risk measure**, coherent if $K = 0$. The premium has to be understood as the expected loss.

Jarrow (2002) :

the put option premium is a measure of the firm insolvency risk (insurance point of view)

\mathcal{R} reduces to the put premium risk measure in Jarrow (2002) when $K = 0$

Dual Representation

The dual representation can be obtained

- either using convex analysis tools
- or **extending** cash sub-additive r.m. to cash invariant r.m. and then exploiting their dual representation

We look for a **“minimal” extension** of \mathcal{R}

Key property :

$$\hat{\rho}(X_T, x) := \mathcal{R}(X_T 1_T - x 1_T) - x \text{ cash invariant as function of } (X_T, x)$$

$$\begin{aligned} \hat{\rho}(X_T + m, x + m) &= \mathcal{R}(X_T + m - (x + m)1_T) - (x + m) \\ &= \hat{\rho}(X_T, x) - m \end{aligned}$$

Enlarged space and filtration

(X_T, x) may be viewed as the “coordinates” of some r.v. \hat{X}_T on the extended space $\hat{\Omega} = \Omega\{1, 0\}$ such that

$$\hat{X}_T(\omega, \theta) = X_T(\omega)\mathbf{1}_{\{\theta=1\}} + x\mathbf{1}_{\{\theta=0\}}$$

The correspondence is one to one.

Extended Risk measure

Let \mathcal{R} be a cash sub-additive risk measure defined on \mathcal{X}_T . \mathcal{R} generates a cash-monetary convex risk-measure $\hat{\rho}$ on the space $\hat{\mathcal{X}}_T$ of the bounded $\hat{\mathcal{F}}_T$ -r.v.

$$\hat{\rho}(X_T\mathbf{1}_{\{\theta=1\}} + x\mathbf{1}_{\{\theta=0\}}) = \hat{\mathcal{R}}(X_T, x) := \mathcal{R}(X_T - x\mathbf{1}_T) - x\mathbf{1}_0,$$

$\hat{\rho}(X_T(\omega)\mathbf{1}_{\{\theta=1\}})$ can be identified with $\mathcal{R}(X_T)$.

$$\hat{\mathcal{R}}(X_T, x) = \mathcal{R}(X_T - x) - x \leq \mathcal{R}(X_T - y) - y \leq \hat{\mathcal{R}}(Y_T, y) - y = \hat{\mathcal{R}}(Y_T, y).$$

In terms of “economic default time”, $\{\theta = 1\}$ may be viewed as $\{T < \tau\}$

Dual representation of \mathcal{R}

Any probability measure $\hat{\mathbb{Q}}$ on $\hat{\Omega}$ can be decomposed into

$$\begin{aligned}\hat{\mathbb{Q}}(X_T \mathbf{1}_{\{\theta=1\}} + x \mathbf{1}_{\{\theta=1\}}) &= \hat{\mathbb{Q}}(\{\theta=1\}) \hat{\mathbb{Q}}(X_T | \{\theta=1\}) + x(1 - \hat{\mathbb{Q}}(\{\theta=1\})) \\ &= \hat{q} \mathbb{Q}^1(X_T) + x(1 - \hat{q})\end{aligned}$$

Characterization The minimal penalty function $\hat{\alpha}(\hat{\mathbb{Q}}) = \alpha(\mu)$ depends only on $\mu = \hat{q} \mathbb{Q}^1$, and

$$\left\{ \begin{array}{l} \hat{\alpha}(\hat{\mathbb{Q}}) = \sup_{X_T \in \mathcal{X}_T, x \in \mathbb{R}} \left\{ \mathbb{E}_{\hat{\mathbb{Q}}}[-(X_T - x) \mathbf{1}_{\theta=1}] - \mathcal{R}(X_T - x) \right\} \\ \quad = \sup_{Y_T \in \mathcal{X}_T} \left\{ \hat{q} \mathbb{Q}^1(-Y_T) - \mathcal{R}(Y_T) \right\} \\ \mathcal{R}(X_T) = \sup_{\mu \in \mathcal{M}_1^s} \left\{ \mu(-X_T) - \alpha(\mu) \right\} \end{array} \right.$$

Remark The risk measure $\hat{\mathcal{R}}$ is not defined on the space \mathcal{X}_T

Convex function and Risk measure on $\{0, 1\}$

Let ρ be a monetary risk measure on $\lambda 1_{\{\theta=1\}} + \mu 1_{\{\theta=0\}}$. Then

$$\rho(\lambda 1_{\{\theta=1\}} + \mu 1_{\{\theta=0\}}) = \rho((\lambda - \mu) 1_{\{\theta=1\}}) - \mu$$

Put $\mathbf{v}(\mathbf{x}) = \rho(\mathbf{x} 1_{\{\theta=1\}})$.

Then v is a real convex decreasing function such that $v(x+m)+m$ is non-decreasing.

The correspondence between v and ρ is one-to one. **Examples :**

$\mathcal{R}^{\bar{\rho}, v}(X_T) := \bar{\rho}(-v(X_T))$ **cash sub-additive r.m.**

- v real, convex, such that $v'_x \in [-1, 0]$ and $v(0) = 0$

- $\bar{\rho}$ cash invariant r.m. on

$\mathcal{R}^{\bar{\rho}, v}(X_T) := \bar{\rho}(-v(X_T))$ composition of risk measures

Remark : the cash invariant r.m. $\hat{\rho}$ is not defined on the space \Rightarrow

Second extension of \mathcal{R} to a cash-invariant r.m. on $\tilde{X} \supset$

Goals :

\Rightarrow to obtain cash-invariant r.m., $\tilde{\rho}$, assessing the risk of $X_T \in$

\Rightarrow given $\tilde{\rho}$, identify \mathcal{R} in terms of

$\mathbf{Q} \in_{1,f} (\mathbf{T})$ and T -measurable discount factors

More natural extension

The aim of this extension is to define a risk measure $\tilde{\rho}$ on a space $\tilde{\mathcal{X}}_T$ of bounded r.v. containing \mathcal{X}_T . We need

\Rightarrow to define the r.v. $\tilde{X}_T = X_T^1 1_{\theta=1} + X_T^0 1_{\theta=0}$

\Rightarrow to introduce a **arbitrary** (normalized) monetary risk measure $\bar{\rho}$ on order to specify a “a priori” risk measure for X_T^0 .

Two monetary risk measures The functional $\tilde{\rho}$ defined by

$$\tilde{\rho}(X_T^1 1_{\theta=1} + X_T^0 1_{\theta=0}) = \mathcal{R}(X_T^1 + \bar{\rho}(X_T^0)) + \bar{\rho}(X_T^0)$$

is a monetary risk measure, with minimal penalty function, for any

$$\tilde{\mathbb{Q}} = \tilde{q}^1 \tilde{\mathbb{Q}}^1 + \tilde{q}^0 \tilde{\mathbb{Q}}^0$$

$$\tilde{\alpha}(\tilde{\mathbb{Q}}) = \alpha_{\mathcal{R}}(\tilde{q}^1 \tilde{\mathbb{Q}}^1) + \tilde{q}^0 \bar{\alpha}(\tilde{\mathbb{Q}}^0)$$

- The functional $\rho_{\mathcal{R}}(X_T) = \tilde{\rho}(X_T, X_T)$ is a monetary risk measure with penalty functional

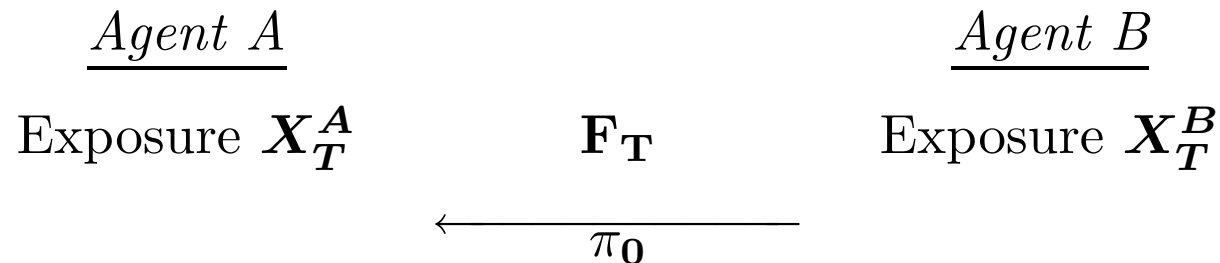
$$\alpha_{\rho_{\mathcal{R}}}(\mathbb{Q}) = \inf_{(\tilde{q}^1, \tilde{\mathbb{Q}}^1, \tilde{q}^0, \tilde{\mathbb{Q}}^0 | \mathbb{Q} = \tilde{q}^1 \tilde{\mathbb{Q}}^1 + \tilde{q}^0 \tilde{\mathbb{Q}}^0)} \alpha_{\mathcal{R}}(\tilde{q}^1 \tilde{\mathbb{Q}}^1) + \tilde{q}^0 \bar{\alpha}(\tilde{\mathbb{Q}}^1)$$

- $\rho_{\mathcal{R}}$ does not allow us to recover \mathcal{R} .
- Let us denote by $D_{0,T}$ the conditional expectation of $1_{\theta=1}$ given \mathcal{F}_T . Then, $\tilde{\mathbb{Q}}(X_T 1_{\theta=1}) = \mathbb{Q}(X_T D_{0,T})$, and

$$\tilde{\mathbb{Q}}(X_T 1_{\theta=0}) = \mathbb{Q}(X_T (1 - D_{0,T})), \Leftrightarrow d\mathbb{Q}^0 = \frac{(1 - D_{0,T})}{\tilde{q}^0} d\mathbb{Q}$$

$$\mathcal{R}(X_t) = \sup_{(\mathbb{Q}, D_{0,T}) \in \mathcal{A}} \{ \mathbb{Q}(-X_T D_{0,T}) - (\alpha_{\mathcal{R}}(D_{0,T} \cdot \mathbb{Q}) + \tilde{q}^0 \bar{\alpha}(\mathbb{Q}^0)) \}$$

Optimal Derivative Design



⇒ We want to determine the optimal transaction (F, π) .

Transaction feasibility

- ★ Agent A looks for a hedge of her exposure : $\inf_{F \in \mathcal{X}, \pi} \mathcal{R}_A(X_T^A - F_T) - \pi_0$.
- ★ Agent B wants to improve her risk measure : $\mathcal{R}_B(X_T^B + F_T) + \pi_0 \leq \rho_B(X_T^B)$.

⇒ **Optimal pricing rule** : $(\pi_B^*)_0(F_T) = \mathcal{R}_B(X_T^B) - \mathcal{R}_B(X_T^B + F)$.

Inf-convolution

Theorem : Let \mathcal{R}_A and \mathcal{R}_B be two cash sublinear risk measures with respective penalty functions α_A and α_B . Let $\mathcal{R}_{A,B}$ be the inf-convolution of \mathcal{R}_A and \mathcal{R}_B

$$\Psi \rightarrow \mathcal{R}_{A,B}(\Psi) \equiv \mathcal{R}_A \square \mathcal{R}_B(\Psi) = \inf_{H \in \mathcal{X}} \{ \mathcal{R}_A(\Psi - H) + \mathcal{R}_B(H) \}$$

and assume that $\mathcal{R}_{A,B}(0) > -\infty$.

- *Then $\rho_{A,B}$ is a cash sub-linear convex risk measure which is finite for all $\Psi \in \mathcal{X}$.*
- *The associated penalty function is given by*

$$\forall \beta, \mu \in \mathcal{M}_{1,f}^s \quad \alpha_{A,B}(\beta, \mu) = \alpha_A(\beta, \mu) + \alpha_B(\beta, \mu).$$

- *$\mathcal{R}_{A,B}$ is continuous from below as soon as this property holds for \mathcal{R}_A or \mathcal{R}_B*

Infinitesimal Risk Measures

⇒ Let $W = (W_t, t \leq T)$ be a Brownian motion on (Ω, \mathcal{F}, P) the natural space associated with W .

⇒ any Q^μ probability equivalent probability measure w.r. to P

$$\frac{dQ^\mu}{dP} := \Gamma_T^\mu, \quad d\Gamma_t^\mu = \Gamma_t^\mu \mu_t^* dW_t, \quad \Gamma_0^\mu = 1$$

⇒ $D_T := e^{-\int_0^T \beta_s ds}$ discount factor ($\beta \geq 0$)

⇒ **penalty functions** : $G(t, \beta, \mu) := \mathbb{E}_{Q^\mu} \left[\int_t^T e^{-\int_t^r \beta_s ds} G(\beta_r, \mu_r) dr \mid \mathcal{F}_t \right]$;

$G(t, \beta, \mu)$ convex w.r. to β and μ and **non increasing** w.r. to β

$$\mathcal{R}_t^g(X_T) = \text{ess sup}_{(\beta, \mu) \in \mathcal{A}} \mathbb{E}_{Q^\mu} \left[\text{cash flow} - \int_t^T e^{-\int_t^r \beta_s ds} G(r, \mu) dr \mid \mathcal{F}_t \right]$$

El Karoui, Peng and Quenez (1997) :

$$-d\mathcal{R}_t^g = g(t, \mathcal{R}_t^g, \mathbf{Z}_t) dt - \langle \mathbf{Z}_t, dW_t \rangle, \quad \mathcal{R}_T^g = -X_T$$

$G(t, \beta\mu)$ is the polar function (Fenchel transform) of $g(t, y, z)$

Summary and conclusions

- ⇒ We introduce a new class of cash sub-additive risk measures \mathcal{R}
- ⇒ \mathcal{R} suitable to assess the risk of financial positions under uncertain interest rates and other risks (e.g. default risk)
- ⇒ we extend \mathcal{R} to cash invariant r.m. defined on enlarged spaces
 - we extend \mathcal{R} to cash invariant r.m. defined on enlarged spaces
 - Dual representation of \mathcal{R} we identify \mathcal{R} from cash invariant r.m. in terms of T -measures and T -stochastic discount factors