Utility based good deal bounds

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Financial Market

$(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$	filtered probability space
S	traded assets (locally bounded semimartingale)
$\mathcal{M}^e(S) \neq \emptyset$	equivalent local martingale measures for S
$\hat{X} \in \mathbf{L}^\infty(\mathcal{F}_T)$	payoff due at time T

Interval of arbitrage free values for $\hat{X}$:

$$\left(\inf_{Q \in \mathcal{M}^e(S)} E_Q[\hat{X}], \sup_{Q \in \mathcal{M}^e(S)} E_Q[\hat{X}] \right)$$

No-good-deal measures

Specify a set $\mathcal{N} \subseteq \mathcal{M}^e(S)$ of “reasonable” valuation measures to obtain a smaller interval of possible values for $\hat{X}$:

$$\left[\inf_{Q\in\mathcal{N}} E_Q[\hat{X}], \sup_{Q\in\mathcal{N}} E_Q[\hat{X}]\right].$$

Possible criterions for “reasonable” valuation measures:

- Chosen as a pricing measures, it does not yield too attractive investment opportunities (good deals).
- Not too far away from the subjective measures P.
- ...

Main difficulty: Convenient definition for $\mathcal{N}$ in a dynamic setting.

Content

- Static good deal bounds;
    - Original definition;
    - Generalized definition;
- Dynamic good deal bounds;
    - Global definition;
    - Lévy framework - local definition;
    - Results and open questions.

Original definition

Sharpe ratio

$$\mathrm{SR}(\hat{X}, Q) := \frac{E[\hat{X} - E_Q[\hat{X}]]}{\sqrt{\mathrm{Var}[\hat{X} - E_Q[\hat{X}]]}}$$

$$\mathrm{SR}(\hat{X}, Q) \leq \sqrt{\mathrm{Var}[Z_T]} = \sqrt{E[Z_T^2] - 1}, \quad Z_T := \frac{dQ}{dP}$$

(Hansen/Jagannathan ’91)

Good deal bounds (Cochrane/Saà-Requejo ’00)

$$\mathcal{N} := \left\{ Q \in \mathcal{M}^e(S) \,\middle|\, Z_T = \frac{dQ}{dP},\ E[Z_T^2] \leq A \right\}$$

$$\left[\inf_{Q \in \mathcal{N}} E_Q[\hat{X}],\ \sup_{Q \in \mathcal{N}} E_Q[\hat{X}] \right]$$

Attainable Payoffs

$$S^Q := \left(S, (E_Q[\hat{X}|\mathcal{F}_t])_{0\le t\le T}\right) \text{ for } Q \in \mathcal{M}^e(S)$$

$$\mathcal{C}(x_0, S^Q) := \left\{ X \in \mathbf{L}^0 \,\middle|\, X \le x_0 + \int_0^T H_s\, dS_s^Q,\ H \in L^a(S^Q),\ X^- \in \mathbf{L}^\infty \right\}$$
$$\subseteq \mathcal{C}(x_0, Q) := \left\{ X \in \mathbf{L}^1(Q) \,\middle|\, E_Q[X] \le x_0, X^- \in \mathbf{L}^\infty \right\}$$

(*Complete market* (S', Q'): $E_{Q'}[X] \le x_0 \iff X \in \mathcal{C}(x_0, S')$.)

Lemma

For $x_0 \in \mathbb{R}$ and $Q \sim P$ with density $Z_T \in \mathbf{L}^2(P)$

$$\sup_{\substack{X \in \mathcal{C}(x_0,Q)\\ E[X]<\infty}} \mathrm{SR}(X, Q) = \sqrt{E[Z_T^2] - 1}.$$

Quadratic utility

Quadratic utility function: $U^q(x) := -(1-x)^2, \;\; x \in \mathbb{R}$.

Lemma

For $x_0 < 1$ and $Q \sim P$ with density $Z_T \in \mathbf{L}^2(P)$

$$\sup_{X \in \mathcal{C}(x_0,Q)} E[U^q(X)] = U^q(x_0)\frac{1}{E[Z_T^2]}.$$

$$\mathcal{N} := \left\{ Q \in \mathcal{M}^e(S) \,\middle|\, Z_T = \frac{dQ}{dP},\; E[Z_T^2] \leq A \right\}$$

Other utility functions

Maximal attainable utility

$U(x)$	$\sup_{X\in\mathcal{C}(x_0,Q)} E[U(X)]$	$f(Q\mid P)$ $(Z_T = dQ/dP)$
$-\frac{1}{\gamma}e^{-\gamma x}, \quad \gamma>0$	$U(x_0)e^{-E[Z_T\log Z_T]}$	$f^e(Q\mid P) := E[Z_T\log Z_T]$
$\log(x)$	$U(x_0) + E[-\log Z_T]$	$f^\ell(Q\mid P) := E[-\log Z_T]$
$\frac{x^\gamma}{\gamma}, \quad 0<\gamma<1,\ x>0$	$U(x_0)E\left[Z_T^{\frac{\gamma}{\gamma-1}}\right]^{1-\gamma}$	$f^p(Q\mid P) := E\left[Z_T^{\frac{\gamma}{\gamma-1}}\right]$

(→ Kramkov/Schachermayer, Schachermayer)

$E[f(Z_T)]$ with f convex is called f-divergence.

Lower bound

No-good-deal measures

We define the set of no-good-deal measures by

$$\mathcal{N} := \{Q \in \mathcal{M}^e(S) \mid f(Q|P) \leq A\}$$

where $A \geq \inf_{Q \in \mathcal{M}^e(S)} f(Q|P)$.

Proposition

Let $u(x_0) := \sup_{X \in \mathcal{C}(x_0,S)} E[U(X)] < \sup_y U(y)$. Then with $u^Q(x_0) := \sup_{X \in \mathcal{C}(x_0,Q)} E[U(X)]$ we have

$$u(x_0) = \inf_{Q \in \mathcal{M}^e(S)} u^Q(x_0).$$

(→ Kramkov/Schachermayer, Schachermayer, Bellini/Frittelli)

Dynamic setting

$$\mathcal{C}_t(x_t,Q) := \left\{ X \in \mathbf{L}^1(Q) \,\middle|\, E_Q[X\mid\mathcal{F}_t] \le x_t \text{ and } X^- \in \mathbf{L}^\infty \right\}$$

$\mathcal{F}_t$-conditional maximal attainable utility

$U(x)$	$\operatorname*{ess\,sup}_{X\in\mathcal{C}_t(x_t,Q)} E[U(X)\mid\mathcal{F}_t]$	$f_t(Q\mid P)$
$-\frac{1}{\gamma}e^{-\gamma x}$	$U(x_t)e^{-E\left[\frac{Z_T}{Z_t}\log\frac{Z_T}{Z_t}\middle\vert\mathcal{F}_t\right]}$	$f_t^e(Q\mid P) := E\left[\frac{Z_T}{Z_t}\log\frac{Z_T}{Z_t}\middle\vert\mathcal{F}_t\right]$
$\log(x)$	$U(x_t)+E\left[-\log\frac{Z_T}{Z_t}\middle\vert\mathcal{F}_t\right]$	$f_t^\ell(Q\mid P) := E\left[-\log\frac{Z_T}{Z_t}\middle\vert\mathcal{F}_t\right]$
$\frac{1}{\gamma}x^\gamma$	$U(x_t)E\left[\left(\frac{Z_T}{Z_t}\right)^{\frac{\gamma}{\gamma-1}}\middle\vert\mathcal{F}_t\right]^{1-\gamma}$	$f_t^p(Q\mid P) := E\left[\left(\frac{Z_T}{Z_t}\right)^{\frac{\gamma}{\gamma-1}}\middle\vert\mathcal{F}_t\right]$
$-(1-x)^2$	$U(x_t)E\left[\left(\frac{Z_T}{Z_t}\right)^2\middle\vert\mathcal{F}_t\right]^{-1}$	$f_t^q(Q\mid P) := E\left[\left(\frac{Z_T}{Z_t}\right)^2\middle\vert\mathcal{F}_t\right]$

No-good-deal measures

No-good-deal measures

Let A' be a process such that $A' \geq 1$. For $t \in [0, T)$ the set of no-good-deal measures is defined by

$$\mathcal{N}_t := \left\{ Q \in \mathcal{M}^e(S) \,\middle|\, f_t(Q|P) \leq A'_t\, f_t(\hat{Q}|P) \right\}.$$

Lemma

$$f_0(\hat{Q}|P) = \inf_{Q \in \mathcal{M}^e(S)} f_0(Q|P) \quad \Rightarrow \quad f_t(\hat{Q}|P) = \operatorname*{ess\,inf}_{Q \in \mathcal{M}^e(S)} f_t(Q|P)$$

(→ Kabanov/Stricker)

Lévy setting

Assumption: $\mathbb{F} = \mathbb{F}^L$ where L is a d-dimensional Lévy process such that $\mathcal{M}^e(S) := \mathcal{M}^e(\mathcal{E}(L)) = \mathcal{M}^e(L) \neq \emptyset$.

Theorem (Girsanov)

$Q \sim P$ has a density process $Z = \mathcal{E}(N^{(\beta,Y)})$ with

$$N_t^{(\beta,Y)} := \int_0^t \beta_s \, dL_s^c + \int_0^t \int_{\mathbb{R}^d} (Y(s,x) - 1) \left(\mu^L(dx,ds) - K(dx)ds \right)$$

and Girsanov parameters β (predictable process) and $Y > 0$ (predictable function).

(→ Jacod/Shiryaev)

Formula for f-divergence

Theorem

For $Q^{(\beta,Y)} \sim P$ with $f(Q^{(\beta,Y)}|P) < \infty$ we have

$$f^e,\ f^\ell\colon\quad f_t(Q^{(\beta,Y)}|P) = E_{R(\beta,Y)}\left[\int_t^T k(\beta_s, Y(s,\,.))\,ds \,\middle|\, \mathcal{F}_t\right]$$

$$f^p,\ f^q\colon\quad f_t(Q^{(\beta,Y)}|P) = E_{R(\beta,Y)}\left[\exp\left(\int_t^T k(\beta_s, Y(s,\,.))\,ds\right) \,\middle|\, \mathcal{F}_t\right]$$

Local definition no-good-deal measures

For a fixed process A with $A \geq 1$ we define

$$\mathcal{NL}_t := \left\{ Q^{(\beta,Y)} \in \mathcal{M}^e(S) \,\middle|\, k(\beta_u, Y_u) \leq A_u k(\hat{\beta}_u, \hat{Y}_u), \forall u \geq t \right\}.$$

Problem

Would like $\mathcal{NL}_t \subseteq \mathcal{N}_t$, i.e.,

$$k(\beta_u, Y_u) \leq A_u k(\hat{\beta}_u, \hat{Y}_u),\ \forall u \geq t \quad \Rightarrow \quad f_t(Q^{(\beta,Y)}|P) \leq A'_t f_t(\hat{Q}|P).$$

Recall: $\mathcal{N}_t = \left\{ Q \in \mathcal{M}^e(S) \,\middle|\, f_t(Q|P) \leq A'_t f_t(\hat{Q}|P) \right\}$

$$f_t(Q^{(\beta,Y)}|P) = E_{R(\beta,Y)}\left[\exp\left(\int_t^T k(\beta_s, Y(s,.))\,ds \right) \,\middle|\, \mathcal{F}_t \right].$$

Theorem

Assume that there exists $Q \in \mathcal{M}^e(S)$ with $f(Q|P) < \infty$.

i) If $f(\hat{Q}|P) = \inf_{Q \in \mathcal{M}^e(S)} f(Q|P)$, then $(\hat{\beta}, \hat{Y})$ are time-independent.

ii) For all $\varepsilon > 0$ there exists $Q^{(\beta,Y)} \in \mathcal{M}^e(S)$ with (β, Y) time-independent and $f(Q^{(\beta,Y)}|P) \leq \inf_{Q \in \mathcal{M}^e(S)} f(Q|P) + \varepsilon$.

(→ with Jeanblanc/Miyahara)

Theorem

If A and $(\hat{\beta}, \hat{Y})$ are time-independent, then

$$\mathcal{NL}_t \subseteq \mathcal{N}_t \quad \text{for all } t \in [0, T].$$

From now on: A and $(\hat{\beta}, \hat{Y})$ are time-independent.

No-good-deal values for $\hat{X}$: $\left[\operatorname*{ess\ inf}_{Q\in\mathcal{NL}_t} E_Q[\hat{X}|\mathcal{F}_t],\ \operatorname*{ess\ sup}_{Q\in\mathcal{NL}_t} E_Q[\hat{X}|\mathcal{F}_t]\right]$

Proposition

$$\operatorname*{ess\ inf}_{Q\in\mathcal{NL}_t} E_Q[\hat{X}|\mathcal{F}_t] = \operatorname*{ess\ inf}_{Q\in\mathcal{NL}_0} E_Q[\hat{X}|\mathcal{F}_t]$$

Lemma

L is also a $Q^{(\beta,Y)}$-Lévy process iff (β, Y) is time-independent.

Proposition

$$\mathcal{NL}_0 \cap \{Q\,|\text{L is a } Q\text{-Lévy process}\} {''}{=}{''} \mathcal{N}_t \cap \{Q\,|\text{L is a } Q\text{-Lévy process}\}.$$

Monetary utility functionals

Definition

$\Phi_t : \mathbf{L}^\infty(\mathcal{F}_T) \to \mathbf{L}^\infty(\mathcal{F}_t)$ is a monetary coherent utility functional at time t if it holds

A) *monotonicity:* $\Phi_t(X_1) \leq \Phi_t(X_2) \quad$ for $X_1 \leq X_2$;

B) *$\mathcal{F}_t$-translation invariance:*
$\Phi_t(X + a_t) = \Phi_t(X) + a_t, \;\; a_t \in \mathbf{L}^\infty(\mathcal{F}_t)$;

C) *concavity:*
$\Phi_t(\alpha X_1 + (1-\alpha)X_2) \geq \alpha\Phi_t(X_1) + (1-\alpha)\Phi_t(X_2), \;\; \alpha \in [0,1]$;

D) *positive homogeneity:* $\Phi_t(\lambda X) = \lambda\Phi_t(X), \quad \lambda \geq 0$.

$\Phi = (\Phi_t(\,.\,))_{0\leq t\leq T}$ is a dynamic monetary coherent utility functional.

m-stable sets

Definition

$\mathcal{R} \subseteq \{Q|Q \sim P\}$ is m-stable if for any $Q^1, Q^2 \in \mathcal{R}$ with density processes Z^1, Z^2 and any stopping time $\tau \leq T$

$$Z_T := Z^1_\tau \frac{Z^2_T}{Z^2_\tau} \quad \text{defines some } Q \in \mathcal{R}.$$

Remark: Let $\tau = t$.

For $G \in \mathbf{L}^\infty(\mathcal{F}_t)$: $\quad E_Q[G|\mathcal{F}_s] = E_{Q^1}[G|\mathcal{F}_s] \quad \forall s \leq t.$

For $G \in \mathbf{L}^\infty$: $\quad E_Q[G|\mathcal{F}_u] = E_{Q^2}[G|\mathcal{F}_u] \quad \forall u \geq t.$

Dynamic consistency

Lower good deal bound: $p_t^\ell(\hat{X}) := \underset{Q\in\mathcal{N}\mathcal{L}_0}{\text{ess inf}}\, E_Q[\hat{X}|\mathcal{F}_t]$

Proposition

- $p^\ell(\hat{X})$ can be derived by dynamic programming techniques.
- $p^\ell(\,.\,)$ is a dynamic monetary coherent utility functional.
- p^ℓ is time-consistent, i.e., if $\sigma \leq \tau$

$$p_\tau^\ell(X) = p_\tau^\ell(Y) \quad \Rightarrow \quad p_\sigma^\ell(X) = p_\sigma^\ell(Y).$$

(→ Delbaen '06)

Remark: $\underset{Q\in\mathcal{N}\mathcal{L}_0}{\text{ess sup}}\, E_Q[\hat{X}|\mathcal{F}_t] = -\underset{Q\in\mathcal{N}\mathcal{L}_0}{\text{ess inf}}\, E_Q[-\hat{X}|\mathcal{F}_t] = -p^\ell(-\hat{X})$

Representation

Assumption: $S = \mathcal{E}(L)$ is continuous.
Superhedging price process for $\hat{X} \in \mathbf{L}^\infty$:

$$\operatorname*{ess\ sup}_{Q \in \mathcal{M}^e(S)} E_Q[\hat{X}|\mathcal{F}_t] = x_0 + \int_0^t H_s\, dS_s - C_t$$

$x_0 \in \mathbb{R}$, C increasing, $\int H\, dS$ is locally bounded from below.

Upper good deal bound: $\operatorname*{ess\ sup}_{Q \in \mathcal{NL}_0} E_Q[X|\mathcal{F}_t] = ?$

Proposition

$\mathcal{NL}_0 = \mathcal{M}^e(\tilde{S})$ for some $\tilde{S}$ iff for each t and ω

$$\left\{ \beta_t(\omega) \in \mathbb{R}^d \,\middle|\, k(\beta_t(\omega)) \le Ak(\hat{\beta}),\ Q^\beta \in \mathcal{M}^e(S) \right\} - \hat{\beta}$$

is a linear space. (→ Delbaen '06)

Let L be locally bounded, $Q^1, Q^2 \in \mathcal{NL}_0$ and define

$$\pi_t(\hat{X}) := \begin{cases} E_{Q^1}[\hat{X}|\mathcal{F}_t] & \text{on } [\![0,\tau[\![\\ E_{Q^2}[\hat{X}|\mathcal{F}_t] & \text{on } [\![\tau, T]\!] \end{cases}$$

Proposition

Define $\overline{Q} \in \mathcal{NL}_0$ by the density $\overline{Z}_T := Z^1_\tau \frac{Z^2_T}{Z^2_\tau}$ and set

$$\Delta^\tau := E_{Q^2}[\hat{X}|\mathcal{F}_\tau] - E_{Q^1}[\hat{X}|\mathcal{F}_\tau].$$

If $N := \Delta^\tau \mathbf{1}_{[\![\tau,T]\!]}$ is a Q^1-martingale, then $\pi(\hat{X})$ is a martingale under $\overline{Q}$. If τ is predictable, the following are equivalent:

- $\pi(\hat{X})$ is a $\overline{Q}$-martingale.
- There exists $Q \in \mathcal{M}^e(S)$ such that $\pi(\hat{X})$ is a Q-martingale.
- $\pi_{\tau-}(\hat{X}) = \pi_\tau(\hat{X})$.

Alternative definition

Alternative definition for good deal bounds:

(→ Jaschke/Küchler, Staum, ...)

Desirable payoffs: $\mathcal{A}_0 := \{X \in \mathbf{L}^\infty | \Phi_0(X) \geq 0\}$ for an MCUF Φ_0
Set of superhegeable payoffs: $\mathcal{C}(0, S)$
Good deal: $X \in \mathcal{C}(0, S)$ such that $X - \varepsilon \in \mathcal{A}_0$ for some $\varepsilon > 0$

Lower good deal bound:

$$\begin{aligned} \pi^{\mathcal{A}_0}(\hat{X}) &= \sup\left\{m \in \mathbb{R} \,\middle|\, \hat{X} - m + X \in \mathcal{A}_0 \text{ for some } X \in \mathcal{C}(0, S)\right\} \\ &= \sup_{X \in \mathcal{C}(0,S)} \Phi_0(\hat{X} + X) \end{aligned}$$

Let $\mathcal{NL}_0 \subseteq \mathcal{M}^e(S)$ be weakly relatively compact and $\Phi_0(\,\cdot\,) = \inf_{Q \in \mathcal{NL}_0} E_Q[\,\cdot\,]$ then

$$\pi^{\mathcal{A}_0}(\hat{X}) = \sup_{X \in \mathcal{C}(0,S)} \Phi_0(\hat{X} + X) = \inf_{Q \in \mathcal{NL}_0} E_Q[\hat{X}].$$

Literature to good deal bounds

- Sharpe ratio: Cochrane/Saà-Requejo ’00, Björk/Slinko ’05
- Indifference Pricing: Černý ’03
- Gain/loss related: Bernardo/Ledoit ’00, Pinar/Salih ’05
- $\mathbf{L}^1$-norm: Longarela ’01

- Monetary utility functionals: Černý/Hodges ’02, Jaschke/Küchler ’01, Staum ’04, Klöppel/Schweizer ’05, Cherny ’05/’06
- Valuation and stress test measures: Carr et al. ’01