Time-consistency of indifference prices and monetary utility functions.

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Setup

$\Omega = \{\omega_1, \ldots, \omega_N\}$ finite sample space.

$\{0, 1, \ldots, T\}$ time set.

$(\mathcal{F}_t)_{t=0}^T$ filtration.
$(\mathcal{F}_0 = \{\emptyset, \Omega\})$

$\mathcal{F}_t$ is generated by the time t atoms $A_t^1, \ldots, A_t^{N_t}$.
$\Omega = A_t^1 \cup \cdots \cup A_t^{N_t}$.

Setup

Assumption: $(\mathcal{F}_t)$ is strictly refining.
(Every time t atom A_t^k splits into at least two parts at time $t+1$.)

$L(\mathcal{F}_t)$ are all $\mathcal{F}_t$-measurable functions.

$L(\mathcal{F}_T)$ are all **uncertain monetary payoffs** at time T.
(interest rate r=0)

Dynamic utility functions

A dynamic utility function $(U_t)_{t=0}^T$ is a family of utility functions $U_t : L(\mathcal{F}_T) \to L(\mathcal{F}_t)$ such that

(LP) Local property: $1_A U_t(X) = 1_A U_t(X 1_A)$
for all $X \in L(\mathcal{F}_T)$, $A \in \mathcal{F}_t$.

(SM) Strict monotonicity: For all $X, Y \in L(\mathcal{F}_T)$,

- $U_t(X) \geq U_t(Y)$ for $X \geq Y$
- $\varepsilon > 0$ and $\omega \in \Omega$,

  $U_t(X + \varepsilon 1_{\{\omega\}}) > U_t(X)$ on the atom A_t^k that contains ω.

Dynamic utility functions

(C) Continuity: U_t is continuous with respect to $||.||_\infty$,
where $||X||_\infty := \max_{\omega \in \Omega} |X(\omega)|$ for $X \in L(\mathcal{F}_T)$.

U_t has the **monetary property (MP)** if

$$U_t(X + m) = U_t(X) + m \quad \text{for all } X \in L(\mathcal{F}_T), m \in L(\mathcal{F}_t).$$

Examples

Fix t, $u : \mathbb{R} \to \mathbb{R}$ continuous, strictly increasing.

1. $U_t(X) = E_P[u(X) \mid \mathcal{F}_t]$ Savage (1954)

2. $U_t(X) = \inf_{P \in \mathcal{Q}} E_P[u(X) \mid \mathcal{F}_t]$ Gilboa-Schmeidler (1989)

3. $U_t(X) = \inf_{P \in \mathcal{Q}} \{E_P[u(X) \mid \mathcal{F}_t] + c(P)\}$.
   plausibility index $c : \mathcal{Q} \to L(\mathcal{F}_t)$.
   Maccheroni, Marinacci, Rustichini (2005)

   $u(x) = x$, U_t has (MP)
   Artzner et al (1999), Föllmer-Schied (2002),
   Frittelli-Rosazza Gianin (2002).

Time-consistency

We call a dynamic utility function **time-consistent** if for all $X, Y \in L(\mathcal{F}_T)$ and $t \in \{0, \dots, T-1\}$,

$$U_{t+1}(X) \geq U_{t+1}(Y) \quad \text{implies} \quad U_t(X) \geq U_t(Y)\,.$$

Examples: $t = 0, \dots, T$

1. $U_t(X) = E_P[u(X) \mid \mathcal{F}_t]$ is time-consistent.

2. $U_t(X) = \inf_{P \in \mathcal{Q}_t} E_P[u(X) \mid \mathcal{F}_t]$
see Epstein-Schneider (2003), Artzner et al. (2003)

3. $U_t(X) = \inf_{P \in \mathcal{Q}_t} \{E_P[u(X) \mid \mathcal{F}_t] + c_t(P)\}$
see Maccheroni, Marinacci, Rustichini (2005)
Detlefsen, Scandolo (2005), Cheridito et al. (2006)

Initial Endowment

Suppose the agent is already holding a portfolio with time T value $V \in L(\mathcal{F}_T)$.

When considering a payoff $X \in L(\mathcal{F}_T)$, then the utility of X at time t is given by

$$U_t^V(X) := U_t(X+V) - U_t(V).$$

(incremental utility).

Certainty equivalents

Definition 1. *The certainty equivalent* $C_t^V(X) \in L(\mathcal{F}_t)$ *of* $X \in L(\mathcal{F}_T)$ *satisfies*

$$U_t^V(C_t^V(X)) = U_t^V(X).$$

C_t^V has the following properties

(1) $1_A C_t^V(X) = 1_A C_t^V(1_A X)\,, \quad$ for all $X \in L(\mathcal{F}_T)$ and $A \in \mathcal{F}_t$.

(2) $C_t^V(m) = m$ for all $m \in L(\mathcal{F}_t)$.

(3) $C_t^V(X) \geq C_t^V(Y) \;\Leftrightarrow\; U_t^V(X) \geq U_t^V(Y)$ for $X, Y \in L(\mathcal{F}_T)$.

(4) Suppose that U_t^V is quasi-concave. Then

$$C_t^V \text{ has the (MP)} \quad \Leftrightarrow \quad C_t^V \text{ is concave.}$$

Indifference prices

The **indifference bid price** $b_t^V(X) \in L(\mathcal{F}_t)$ of $X \in L(\mathcal{F}_T)$ with respect to U_t^V is given by

$$U_t^V(X - b_t^V(X)) = 0\,.$$

$b_t^V(X)$ satisfies

$$U_t(V + X - b_t^V(X)) = U_t(V)\,.$$

The ask price $a_t^V(X) = -b_t^V(-X)$ satisfies the equation

$$U_t(V - X + a_t^V(X)) = U_t(V).$$

$$b_t^V(X) = -C_t^{V+X}(-X) \quad \text{and} \quad a_t^V(X) = C_t^{V-X}(X)\,.$$

Time-consistency of indifference prices

$(b_t^V)_{t=1}^T$ is a monetary utility function. It satisfies (LP), (SM), (C) and (MP).

Monetary property: $b_t^V(X+m) = b_t^V(X) + m$ for $X \in L(\mathcal{F}_T)$, $m \in L(\mathcal{F}_t)$.

The family of bid prices b_t^V is time-consistent if for all $X, Y \in L(\mathcal{F}_T)$ and $t \in \{0, \ldots, T-1\}$,

$$b_{t+1}^V(X) \geq b_{t+1}^V(Y) \quad \text{implies} \quad b_t^V(X) \geq b_t^V(Y)\,.$$

Main Result

Theorem 2. *Suppose that* $(U_t)_{t=0}^T$ *is a time-consistent dynamic utility function. Then the following are equivalent:*

(1) C_t^V *satisfies (MP) for all* $t \in \{1, \dots, T\}$ *and all* $V \in L(\mathcal{F}_T)$*;*

(2) $(b_t^V)_{t=0}^T$ *is time-consistent for all* $V \in L(\mathcal{F}_T)$*.*

(1) ⇒ (2): see Musiela-Zariphopoulou (2004)
Klöppel-Schweizer (2005), Barrieu and El Karoui (2005)
Leitner (2006)

Indifference sets

For $0 \leq s \leq t \leq T$ and $V \in L(\mathcal{F}_T)$, we define the **indifference set** $\mathcal{I}_{s,t}^V$ by

$$\begin{aligned}
\mathcal{I}_{s,t}^V &:= \{X \in L(\mathcal{F}_t) \mid U_s(V+X) = U_s(V)\} \\
&= \left\{X \in L(\mathcal{F}_t) \mid U_s^V(X) = 0\right\} \\
&= \left\{X \in L(\mathcal{F}_t) \mid C_s^V(X) = 0\right\} \\
&= \left\{X \in L(\mathcal{F}_t) \mid b_s^V(X) = 0\right\}.
\end{aligned}$$

Decomposition Theorem

Theorem 3. *The following are equivalent:*

(1) $(U_t)_{t=0}^T$ *is time-consistent;*

(2) $(U_t^V)_{t=0}^T$ *is time-consistent for all* $V \in L(\mathcal{F}_T)$*;*

(3) $(C_t^V)_{t=0}^T$ *is time-consistent for all* $V \in L(\mathcal{F}_T)$*;*

(4) $C_s^V(X) = C_s^V(C_t^V(X))$ *for all* $s \leq t$*,* $V, X \in L(\mathcal{F}_T)$*;*

(5) $\mathcal{I}_{s,T}^V = \bigcup_{Y \in \mathcal{I}_{s,t}^V} (Y + \mathcal{I}_{t,T}^{V+Y})$ *for all* $s \leq t$*,* $V \in L(\mathcal{F}_T)$*.*

Example: Expected Utility

Let $u : \mathbb{R} \to \mathbb{R}$ be continuous and strictly increasing. P a strictly positive probability measure on Ω.

$$U_t(X) = E_P[u(X) \mid \mathcal{F}_t]\,, \; t = 0, \dots, T, \quad \text{is time-consistent.}$$

The indifference prices $(b_t^V)_{t=0}^T$ are time-consistent
$\Leftrightarrow$ $C_t^0(X) = u^{-1} E_P[u(X) \mid \mathcal{F}_t]$ is monetary for all $t \geq 1$
$\Leftrightarrow$ CARA
$\Leftrightarrow$

$$u(x) = \begin{cases} a + bx & a \in \mathbb{R}, b > 0 \\ a - b\,\text{exp}(-\gamma x) & a \in \mathbb{R}, b, \gamma \geq 0 \end{cases} .$$

Part II: Dynamic monetary risk measures for stochastic processes

$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^{T}, P)$ filtered probability space

$\{0, \ldots, T\}$ discrete time set.

$\mathcal{R}^\infty$ denotes the set of all bounded adapted stochastic processes on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^{T}, P)$.

The risky objects are stochastic processes $X \in \mathcal{R}^\infty$ modelling discounted value processes or discounted cumulated cash flows.

Monetary utility functions on $\mathcal{R}^\infty$

$\pi_{t,s} : \mathcal{R}^\infty \to \mathcal{R}^\infty \,, \quad \pi_{t,s}(X)_r := 1_{\{t \le r\}} X_{r \wedge s} \quad r = 0, \dots, T$

Definition: A concave monetary utility function on $\mathcal{R}^\infty_{t,T}$ is a mapping $\phi_t : \mathcal{R}^\infty_{t,T} \to L^\infty(\mathcal{F}_t)$ such that

(N) $\phi(0) = 0$

(M) $\phi_t(X) \ge \phi_t(Y)$ for $X \ge Y$, $X, Y \in \mathcal{R}^\infty_{t,T}$

(MP) $\phi_t(X + m 1_{[t,T]}) = \phi_t(X) + m$, $X \in \mathcal{R}^\infty_{t,T}$ and $m \in L^\infty(\mathcal{F}_t)$

(C) $\phi_t(\lambda X + (1-\lambda) Y) \ge \lambda \phi_t(X) + (1-\lambda)\phi_t(Y)$ for all $X, Y \in \mathcal{R}^\infty_{t,T}$ and $\lambda \in L^\infty(\mathcal{F}_t)$.

Dynamic monetary utility functions on $\mathcal{R}^\infty$

A dynamic concave monetary utility function on $\mathcal{R}^\infty$ is a family $(\phi_t)_{t=0}^T$ of concave monetary utility functions on $\mathcal{R}^\infty_{t,T}$.

Definition: $(\phi_t)_{t=0}^T$ is time-consistent if

$$\phi_t(X) = \phi_t(X_t 1_{\{t\}} + \phi_{t+1}(X) 1_{[t+1,T]}) \quad \forall X \in \mathcal{R}^\infty,\ t = 0, \dots, T-1$$

Define the aggregator $G_t : L^\infty(\mathcal{F}_t) \times L^\infty(\mathcal{F}_{t+1}) \to L^\infty(\mathcal{F}_t)$ as

$$G_t(X_t, X_{t+1}) := \phi_t(X_t, X_{t+1}, \dots, X_{t+1}).$$

Aggregator

G_t satisfies

(G1) $G_t(0,0) = 0$

(G2) $G_t(X_t, X_{t+1}) \geq G_t(Y_t, Y_{t+1})$ if $X_t \geq Y_t$ and $X_{t+1} \geq Y_{t+1}$

(G3) $G_t(X_t + m, X_{t+1} + m) = G_t(X_t, X_{t+1}) + m$ for $m \in L^\infty(\mathcal{F}_t)$

(G4) G_t is concave.

By backward induction:
$\phi_T(X) = X_T$
$\phi_t(X) = G_t(X_t, \phi_{t+1}(X))$, $t \leq T - 1$.

Generator

The mapping $H_t : L^\infty(\mathcal{F}_{t+1}) \to L^\infty(\mathcal{F}_t)$ given by

$$H_t(X) := G_t(0, X)$$

satisfies

(H1) $H_t(0) = 0$

(H2) $H_t(X) \geq H_t(Y)$ if $X \geq Y$

(H3) $H_t(X + m) \leq H_t(X) + m$ for $m \in L^\infty_+(\mathcal{F}_t)$.

(H4) H_t is concave.

Representation result

Let $\mathcal{E}_t := \Big\{\xi \in L^1_+(\mathcal{F}_t) \mid E_P[\xi \mid \mathcal{F}_{t-1}] \leq 1\Big\}$. Every sequence $(\xi_{t+1}, \dots, \xi_T) \in \mathcal{E}_{t+1} \times \cdots \times \mathcal{E}_T$ induces a P-supermartingale $(M_r^\xi)_{r=0}^T$ by

$$M_r^\xi := \begin{cases} 1 & \text{for } r \leq t \\ \xi_{t+1} \cdot \dots \cdot \xi_r & \text{for } r = t+1, \dots, T \end{cases}$$

Theorem: Suppose that

$$H_t(X) := \operatorname*{ess\,inf}_{\xi_{t+1} \in \mathcal{E}_{t+1}} \Big\{E_P[\xi_{t+1} X \mid \mathcal{F}_t] + \psi_t(\xi_{t+1})\Big\}$$

Then,

$$\phi_t(X) = X_t + \operatorname*{ess\,inf}_{(\xi_{t+1},\dots,\xi_T) \in \mathcal{E}_{t+1} \times \cdots \times \mathcal{E}_T} E_P\Big[\sum_{j=t+1}^{T} M_j^\xi \Delta X_j + M_{j-1}^\xi \psi_{j-1}(\xi_j) \mid \mathcal{F}_t\Big].$$