Dynamic Variational Preferences and Monetary Concave Utility Functions

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ELLSBERG PARADOX

An urn contains 90 balls; of these,
30 are Red, and the remaining
60 are Blue or Green, in unspecified proportions.
For several individuals:

$$\mathbf{10}_R \succ \mathbf{10}_B, \tag{1}$$

but

$$\mathbf{10}_{R\cup G} \prec \mathbf{10}_{B\cup G}. \tag{2}$$

There cannot exists a probability p on $\{R, B, G\}$ representing their beliefs. Per contra:

(1) implies $p(R) > p(B)$,

(2) implies $p(R \cup G) < p(B \cup G)$ i.e. $p(R) + p(G) < p(B) + p(G)$.

Agents dislike ambiguity

MODEL UNCERTAINTY

Gilboa and Schmeidler (1989) provide a behavioral characterization of agents who rank payoff profiles h according to

$$V(h) = \inf_{p \in C} \mathsf{E}^p [u(h)], \tag{MP}$$

for some closed convex set C of probability measures.

Hansen and Sargent (2001) consider agents who rank payoff profiles h according to

$$V(h) = \inf_{p \ll q} \left(\mathsf{E}^p [u(h)] + \theta R(p \| q) \right), \tag{EP}$$

where R is the relative entropy of p w.r.t. q.

VARIATIONAL PREFERENCES

MMR (2004) provide a behavioral characterization of agents who rank payoff profiles h according to

$$V(h) = \inf_p \left(\mathsf{E}^p \left[u(h) \right] + c(p) \right), \tag{VP}$$

for some closed convex function c on probability measures.

MP is the special case of VP in which c only takes values 0 and ∞.

EP is the special case of VP in which c is proportional to $R(\cdot||q)$.

RISK NEUTRALITY AND MCUFS

Under risk neutrality, i.e. $u(t) = t$ for all $t \in \mathbb{R}$, V is a *Monetary Concave Utility Function* (a *Concave Niveloid*, the opposite of a *Convex Risk Measure*):

Ben-Tal (1985), Dolecki and Greco (1991), Föllmer and Schied (2002), Frittelli and Rosazza (2002), Barrieu and El Karoui (2004), Bion-Nadal (2004), Dana (2005), Detlefsen and Scandolo (2005), Filipovic and Kupper (2005), Cheridito and Kupper (2006), Cheridito, Delbaen and Kupper (2006), Klöppel and Schweizer (2006).

MALEVOLENT NATURE

The interpretation of Hart, Modica and Schmeidler (1994) of MP:

"... the uncertainty averse decision maker behaves 'as if' there were an opponent [Nature] who could partially influence occurrence of states to his disadvantage ...

... *think of the opponent as choosing* p *in* C ..."

holds unchanged for VP:

... *think of* $c(p)$ *as the opponent's cost of choosing* p ...

Thus $c(p)$ reflects the relative plausibility attributed to p by the agent.

This interpretation finds support in recent experimental findings (e.g. Hsu, Bhatt, Adolphs, Tranel and Camerer, 2005 and Rustichini, 2005).

A DYNAMIC SETUP

Macroeconomic and Financial applications of VP are naturally set in dynamic environments in which VP roughly takes the form

$$V_t(h) = \inf_p \left(\mathsf{E}^p \left[\sum_{\tau \geq t} \beta^{\tau - t} u(h_\tau) \,|\mathcal{G}_t \right] + c_t(p\,|\mathcal{G}_t) \right) \tag{DVP}$$

where β is a discount factor and $\mathcal{G}_t$ represents the information available at time t.

Moreover, recursivity is often required for analytical tractability.

In this work, we provide a behavioral foundation for DVP and a characterization of their recursivity.

Epstein and Schneider (2003) obtained similar results for the special MP case.

INFORMATION AND CONSUMPTION

$\mathcal{T} = \{0, 1, ..., T\}$.

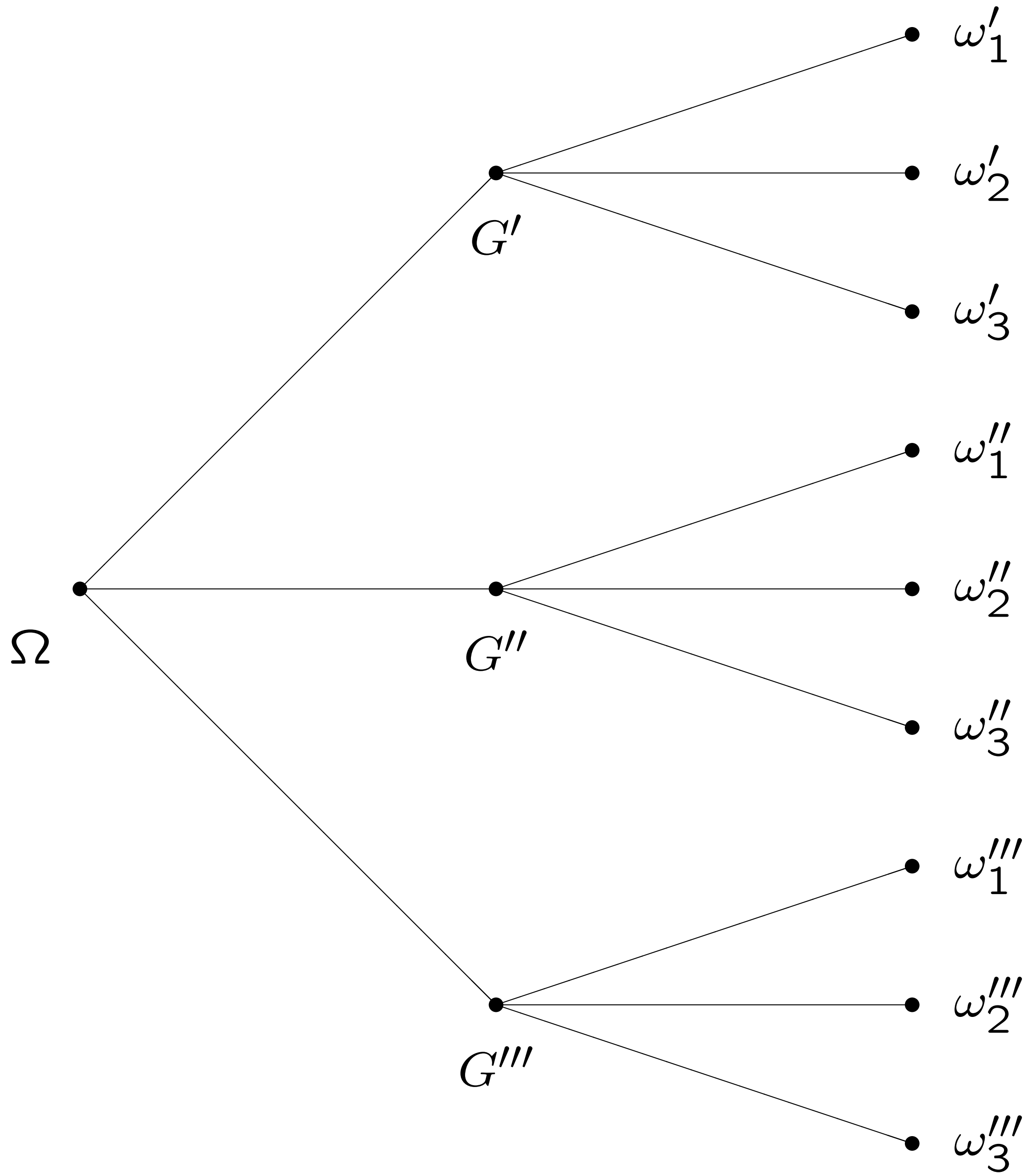
Information is an event tree $\{\mathcal{G}_t\}_{t \in \mathcal{T}}$ defined on a finite space Ω.

$G_t(\omega)$ is the element of $\mathcal{G}_t$ that contains ω.

Consumption lies in a convex set X.

Agents choose among *acts*: X-valued adapted processes $h = (h_t)_{t \in \mathcal{T}}$.

$y = (y_t)_{t \in \mathcal{T}}$ in $X^{\mathcal{T}}$ represent *certain consumption streams*.


ω_1'
ω_2'
G'
ω_3'
ω_1''
Ω
ω_2''
G''
ω_3''
ω_1'''
ω_2'''
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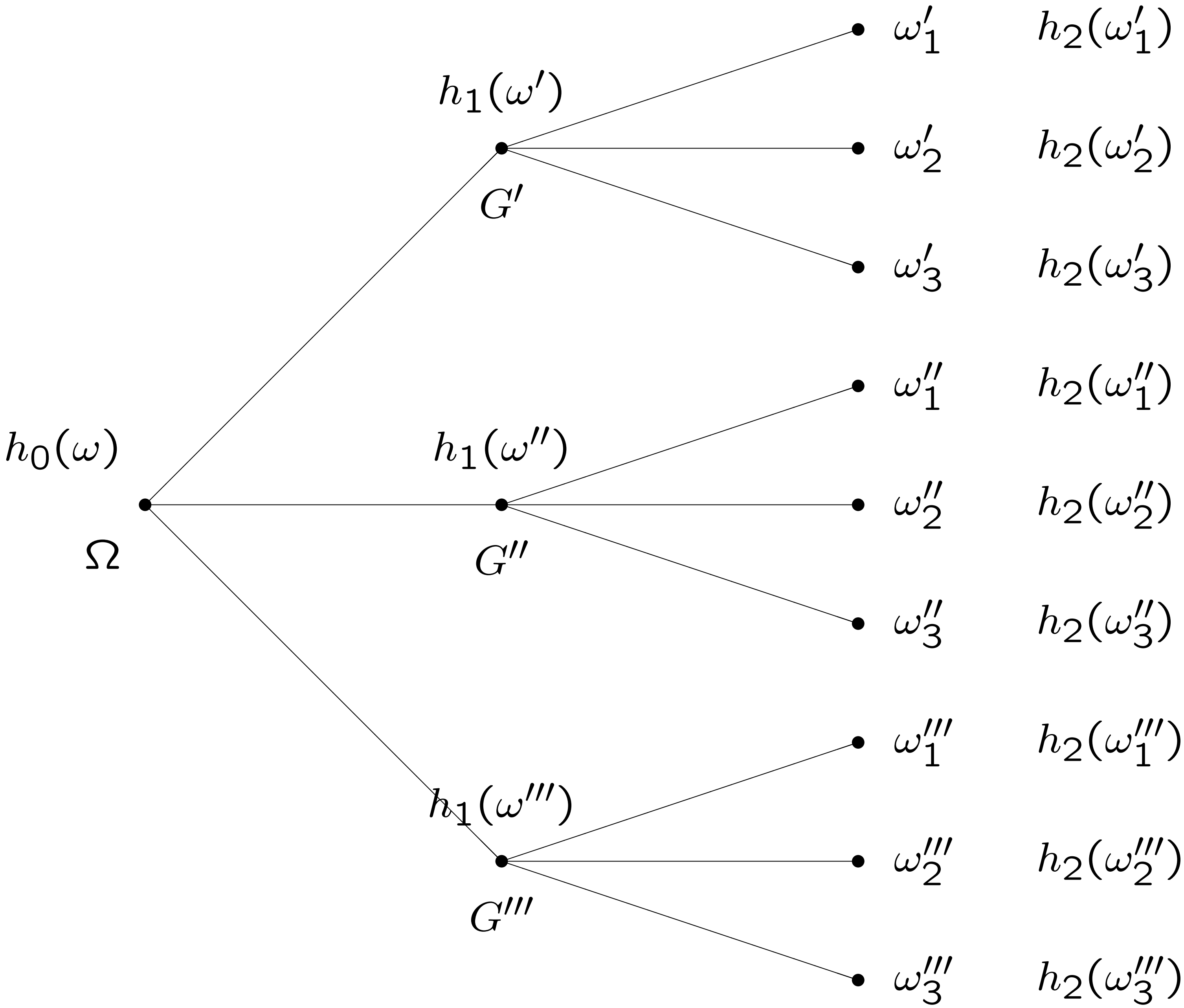
ω_3'''



$h_0(\omega)$
Ω
$h_1(\omega')$
G'
$h_1(\omega'')$
G''
$h_1(\omega''')$
G'''
ω'_1 $h_2(\omega'_1)$
ω'_2 $h_2(\omega'_2)$
ω'_3 $h_2(\omega'_3)$
ω''_1 $h_2(\omega''_1)$
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PREFERENCES

Let the binary relations $\left\{\succsim_{t,\omega}\right\}$ on the set of all acts represent the decision maker's preferences at any time-state node (t, ω).

Axiom 1 (Conditional preference—CP) *For each* (t, ω)*:*

(i) $\succsim_{t,\omega}$ *coincides with* $\succsim_{t,\omega'}$ *if* $G_t(\omega) = G_t(\omega')$.

(ii) If $h(\tau, \omega') = h'(\tau, \omega')$ *for all* $\tau \geq t$ *and* ω' *in* $G_t(\omega)$*, then* $h \sim_{t,\omega} h'$.

Axiom 2 (Variational preferences—VP) *For each* (t, ω)*:*

(i) $\succsim_{t,\omega}$ *is complete and transitive.*

(ii) Let h, h' *be acts,* y, y' *certain consumption streams, and* α *in* $(0, 1)$*, if* $\alpha h + (1-\alpha) y \succsim_{t,\omega} \alpha h' + (1-\alpha) y$ *then* $\alpha h + (1 - \alpha) y' \succsim_{t,\omega} \alpha h' + (1 - \alpha) y'$*.*

(iii) For all h, h', h''*, the sets* $\{\alpha \in [0, 1] : \alpha h + (1 - \alpha) h' \succsim_{t,\omega} h''\}$ *and* $\{\alpha \in [0, 1] : h'' \succsim_{t,\omega} \alpha h + (1 - \alpha) h'\}$ *are closed.*

(iv) If $h(\omega') \succsim_{t,\omega} h'(\omega')$ *for all* ω'*, then* $h \succsim_{t,\omega} h'$*.*

(v) If $h \sim_{t,\omega} h'$*, then* $\alpha h + (1 - \alpha) h' \succsim_{t,\omega} h$ *for all* α *in* $(0, 1)$*.*

Axiom 3 (Risk preference—RP) *Let* y, y' *in* $X^{\mathcal{T}}$*:*

(i) For all x, x', x'', x''' *in* X*, if* $\left(y_{-\{\tau,\tau+1\}}, x, x'\right) \succsim_{t,\omega} \left(y_{-\{\tau,\tau+1\}}, x'', x'''\right)$ *holds for some* (t, ω) *and some* $\tau \geq t$*, then it holds for all* (t, ω) *and all* $\tau \geq t$*.*

(ii) For each (t, ω)*, there exist* $x \succ_{t,\omega} x'$ *in* X *such that for all* α *in* $(0, 1)$ *there is* x'' *in* X *satisfying either* $x' \succ_{t,\omega} \alpha x'' + (1 - \alpha) x$ *or* $\alpha x'' + (1 - \alpha) x' \succ_{t,\omega} x$*.*

Axiom 4 (Dynamic consistency—DC) *For each* (t, ω) *with* $t < T$, *if* $h_\tau = h'_\tau$ *for all* $\tau \leq t$ *and* $h \succsim_{t+1,\omega'} h'$ *for all* ω', *then* $h \succsim_{t,\omega} h'$.

A state ω'' is $\succsim_{t,\omega}$-null if

$$h\left(\omega'\right) = h'\left(\omega'\right) \text{ for all } \omega' \neq \omega'' \text{ implies } h \sim_{t,\omega} h'.$$

Axiom 5 (Full support—FS) *For each* (t, ω), ω *is not* $\succsim_{t,\omega}$*-null.*

(Under CP and DC this is equivalent to the weaker: *No* ω *is* $\succsim_0$*-null*).

DYNAMIC AMBIGUITY INDEXES

If $E \subseteq \Omega$, $\Delta(E)$ (resp. $\Delta^{++}(E)$) is the set of probability measures on Ω with support contained in E (resp. equal to E).

A *dynamic ambiguity index* is a family $\{c_t\}_{t \in \mathcal{T}}$ of non-negative functions $c_t : \Omega \times \Delta(\Omega) \to [0, \infty]$ such that for all t in $\mathcal{T}$:

(i) $c_t(\cdot, p) : \Omega \to [0, \infty]$ is $\mathcal{G}_t$-measurable for all p in $\Delta(\Omega)$,

(ii) $c_t(\omega, \cdot) : \Delta(\Omega) \to [0, \infty]$ is grounded, closed and convex, with $\operatorname{dom} c_t(\omega, \cdot) \subseteq \Delta(G_t(\omega))$ and $\operatorname{dom} c_t(\omega, \cdot) \cap \Delta^{++}(G_t(\omega)) \neq \varnothing$, for all ω in Ω.

For each (t, ω, p), $p_t(\omega)$ denotes the conditional probability $p(\cdot\,|\mathcal{G}_t)(\omega)$, and $p_{|\mathcal{G}_{t+1}}$ the restriction of p to $\mathcal{G}_{t+1}$.

DYNAMIC VARIATIONAL PREFERENCES

Theorem 1 *The following statements are equivalent:*

(a) $\left\{\succsim_{t,\omega}\right\}$ *satisfy CP, VP, RP, and FS.*

(b) There exist $\beta > 0$*, an unbounded affine function* $u : X \to \mathbb{R}$*, and a dynamic ambiguity index* $\{c_t\}$ *such that, for each* (t, ω)*,* $\succsim_{t,\omega}$ *is represented by*

$$V_t(\omega, h) = \inf_{p \in \Delta^{++}(\Omega)} \left(\mathsf{E}^{p_t(\omega)} \left[\sum_{\tau \geq t} \beta^{\tau - t} u(h_\tau) \right] + c_t(\omega, p_t(\omega)) \right)$$

for all h. *(DVP)*

Moreover, $\left(\tilde{\beta}, \tilde{u}, \{\tilde{c}_t\}\right)$ *represent* $\succsim_{t,\omega}$ *in the sense of DVP if and only if* $\tilde{\beta} = \beta$*,* $\tilde{u} = au + b$ *for some* $a > 0$ *and* $b \in \mathbb{R}$*, and* $\{\tilde{c}_t\} = \{ac_t\}$*.*

DYNAMIC CONSISTENCY

Theorem 2 *A DVP satisfies DC iff*

$$c_t(\omega, q) = \beta \mathsf{E}^{q_{|\mathcal{G}_{t+1}}}\left[c_{t+1}\left(\cdot, q_{t+1}(\cdot)\right)\right] + \inf_{p_{|\mathcal{G}_{t+1}} = q_{|\mathcal{G}_{t+1}}} c_t(\omega, p) \qquad \text{(R)}$$

for all ω*, all* $t < T$*, and all* q *in* $\Delta(G_t(\omega))$*.*

The cost for Nature of choosing q at time t in state ω can be decomposed as the sum of: the discounted expected cost of choosing q's conditionals at time $t+1$, plus the minimum cost of inducing $q_{|\mathcal{G}_{t+1}}$ as one period ahead marginal.

Detlefsen and Scandolo (2005) and Cheridito, Delbaen and Kupper (2006) obtain conditions related to (R) in studying the time consistency of risk measures.

RECURSIVITY

Dynamic ambiguity indexes satisfying (R) for some $\beta > 0$ are called *recursive ambiguity indexes*. In fact, they imply the following recursive formulation

$$V_t(\omega, h) = u(h_t(\omega)) + \inf_{r \in \Delta(\Omega, \mathcal{G}_{t+1})} \left(\beta \mathsf{E}^r \left[V_{t+1}(h) \right] + \gamma_t(\omega, r) \right) \quad \text{(RVP)}$$

of the agent's preference functional V_t, where

$$\gamma_t(\omega, r) = \inf_{p \in \Delta(G_t(\omega)) : p_{|\mathcal{G}_{t+1}} = r} c_t(\omega, p) \quad \text{(OPA)}$$

for all $r \in \Delta\left(\Omega, \mathcal{G}_{t+1}\right)$.

ONE PERIOD AHEAD AMBIGUITY INDEXES

Proposition 3 *Let $\{c_t\}$ be a dynamic ambiguity index, and for all $t < T$ define γ_t by (OPA):*

The family $\{\gamma_t\}_{t<T}$ of functions $\gamma_t : \Omega \times \Delta\left(\Omega, \mathcal{G}_{t+1}\right) \to [0, \infty]$ is such that for all $t < T$:

(i) $\gamma_t(\cdot, r) : \Omega \to [0, \infty]$ *is $\mathcal{G}_t$-measurable for all $r \in \Delta\left(\Omega, \mathcal{G}_{t+1}\right)$.*

(ii) $\gamma_t(\omega, \cdot) : \Delta\left(\Omega, \mathcal{G}_{t+1}\right) \to [0, \infty]$ *is grounded, closed, and convex,* $\operatorname{dom} \gamma_t(\omega, \cdot) \subseteq \Delta\left(G_t(\omega), \mathcal{G}_{t+1}\right)$, *and* $\operatorname{dom} \gamma_t(\omega, \cdot) \cap \Delta^{++}\left(G_t(\omega), \mathcal{G}_{t+1}\right) \neq \varnothing$, *for all $\omega \in \Omega$.*

Call *one-period-ahead ambiguity index* a family $\{\gamma_t\}_{t<T}$ of functions that satisfies conditions (i) and (ii).

Theorem 4 *Let* $\{c_t\}_{t \in \mathcal{T}}$ *be a family of functions from* $\Omega \times \Delta(\Omega)$ *to* $[0, \infty]$*. The following statements are equivalent:*

(a) $\{c_t\}$ *is a recursive ambiguity index.*

(b) There exist $\beta > 0$ *and a one-period-ahead ambiguity index* $\{\gamma_t\}$ *such that, for all* $\omega \in \Omega$*,*

$$c_T(\omega, \cdot) = \delta_{\{d_\omega\}}(\cdot), \text{ and for all } t < T$$

$$c_t(\omega, q) = \begin{cases} \beta \mathsf{E}^{q_{|\mathcal{G}_{t+1}}}\left[c_{t+1}\left(\cdot, q_{t+1}(\cdot)\right)\right] + \gamma_t\left(\omega, q_{|\mathcal{G}_{t+1}}\right) & \text{if } q \in \Delta(G_t(\omega)) \\ \infty & \text{else.} \end{cases}$$

In this case, $\{\gamma_t\}$ *is unique and satisfies (OPA).*

See also Cheridito and Kupper (2006).

SUPERDIFFERENTIALS

Let $X = \Delta(\mathbb{R})$ and $\mathcal{F}$ be the subset of $\mathcal{H}$ consisting of monetary (i.e., real valued) acts. Consider a recursive variational preference functional $V_t(\omega, \cdot) : \mathcal{H} \to \mathbb{R}$, under the assumption that the associated utility $u : X \to \mathbb{R}$ is concave (thus reflecting risk aversion) and strictly increasing on $\mathbb{R}$.

Like Epstein and Wang (1994), for $\omega \in \Omega$, $t < T$, and $f \in \mathcal{F}$, we call *one-period-ahead directional derivative* of $V_t(\omega, \cdot)$ at f the functional $V_t'(\omega, f; \cdot) : \mathcal{E}^t \to \mathbb{R}$ defined by

$$V_t'(\omega, f; e) = \lim_{\lambda \downarrow 0} \frac{V_t(\omega, f + \lambda e) - V_t(\omega, f)}{\lambda} \quad \forall e \in \mathcal{E}^t,$$

where $\mathcal{E}^t$ is the subspace of $\mathcal{F}$ consisting of all processes e such that $e_\tau = 0$ if $\tau \neq t, t+1$. These processes represent current and one-period-ahead consumption perturbations.

Denote by $\mathbb{M}\left(G_t(\omega), \mathcal{G}_{t+1}\right)$ the set of all measures on the algebra generated by $\mathcal{G}_{t+1}$ that vanish on each subset of $G_t(\omega)^c$. A *one-period-ahead supergradient* of $V_t(\omega, \cdot)$ at f is an element (k, m) of $\mathbb{R} \times \mathbb{M}\left(G_t(\omega), \mathcal{G}_{t+1}\right)$ such that

$$V_t'(\omega, f; e) \leq k e_t(\omega) + \beta \int e_{t+1} dm, \qquad \forall e \in \mathcal{E}^t.$$

The *one-period-ahead superdifferential* $\partial V_t(\omega, f)$ of $V_t(\omega, \cdot)$ at f is the set of all one-period-ahead supergradients at f.

In particular, $V_t(\omega, \cdot)$ is (one-period-ahead Gateaux) differentiable at f if and only if $\partial V_t(\omega, f)$ is a singleton; in this case, $\partial V_t(\omega, f) = \{V_t'(\omega, f; \cdot)\}$.

A CHAIN RULE

Theorem 5 *For all* $\omega \in \Omega$*,* $t < T$*, and* $f \in \mathcal{F}$*,* $\partial V_t(\omega, f)$ *consists of all pairs*

$$\left(u'(f_t(\omega)), u'\left(f_{t+1}\right) d\rho\right) \qquad \text{(SUP)}$$

such that $u'(f_t(\omega)) \in \partial u(f_t(\omega))$*,* $u'\left(f_{t+1}\right)$ *is a* $\mathcal{G}_{t+1}$*-measurable selection of* $\partial u\left(f_{t+1}\right)$*, and* $\rho \in \arg\inf_{r \in \Delta(\Omega, \mathcal{G}_{t+1})} \left(\beta \mathsf{E}^r \left[V_{t+1}(f)\right] + \gamma_t(\omega, r)\right)$*.*

In particular, $V_t(\omega, \cdot)$ *is differentiable on* $\mathcal{F}$ *if and only if* u *is differentiable on* $\mathbb{R}$ *and* $\gamma_t(\omega, \cdot)$ *is essentially strictly convex.*

This theorem generalizes a result of Epstein and Wang (1994) for recursive MP preferences, and it can be used to extend their asset pricing analysis to recursive variational preferences (work in progress).

DYNAMIC MULTIPLIER PREFERENCES

E.g., if u is differentiable on $\mathbb{R}$ and $c_t(\cdot, p) = \beta^{-t}\theta R(p_t(\cdot) \| q_t(\cdot))$, then $V_t(\omega, \cdot)$ is differentiable and formula (SUP) takes the following neat form:

$$V_t'(\omega, f; e) = u'(f_t(\omega))\, e_t(\omega) + \beta \frac{\mathsf{E}^{q_t(\omega)}\left[e_{t+1} u'\left(f_{t+1}\right) \exp\left(-\frac{\beta^{t+1}}{\theta} V_{t+1}(f)\right)\right]}{\mathsf{E}^{q_t(\omega)}\left[\exp\left(-\frac{\beta^{t+1}}{\theta} V_{t+1}(f)\right)\right]}$$

for each $f \in \mathcal{F}$ and $e \in \mathcal{E}^t$.