G-Expectation, G-Measure of Risk and Related Stochastic Calculus

Shige PENG
Institute of Mathematics, Shandong University250100, Jinan,

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- 1–dim. standard G-normal distribution and main properties.

- 1–dim. G-Brownian motion, G–expectation: main properties.
- Stochastic integral of G-BM,

quadratic variation processes

and Itô's formula

- The existence and uniqueness of SDE
- Dynamic risk measures.

[Peng1997] g–expectation

see [Book: BSDE, El Karoui & Mazliak]

[Peng1999]: Nonlinear Doob-Meyer THM

[Chen-Peng1998–2000]

[Briand-Coquet-Hu-Memin-Peng2000–2003]

[Chen-Epstein2002] Ambiguity by g-expectation

[Rosazza Gianin2003] Dynamic risk measure by g-exp.

[Barrieu-El Karoui2004] Risk measure by g-exp.

[Klöppel–Schweizer2005] utility valuation

[Delbean-Peng-Rosazza Gianin2006]

[Peng2006] Statistical test of CME's market maker's dynamic risk-measures by g–exp. criteria

[Peng Xu2006] g_Γ–Epectation

[Lyons1995AMF] Uncertain volatility and the risk free synthesis of derivatives.

Recall: a classical standard normal distribution $\xi \sim N(0,1)$:

$$E[\phi(\xi)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \phi(y) dy, \quad \forall \phi \in lip(\mathbb{R})$$

or $E[\phi(\xi)] = u(1,0)$

$\partial_t u - \frac{1}{2}\partial^2_{xx} u = 0, \quad u(0,x) = \phi(x).$

$$u(t,x) = E[\phi(x+\sqrt{t}\xi)], \ \forall \phi \in lip(\mathbb{R})$$

1 G–normal distributions

Def. A real valued random variable ξ with the standard G–normal distribution is defined by its G–expectation:

$$\mathbb{E}[\phi(\xi)] := u(1,0), \quad \forall \phi \in lip(\mathbb{R})$$

where $u = u(t,x)$: the solution of the nonlinear heat equation

$$\partial_t u - \frac{1}{2}[(\partial_{xx}^2 u)^+ - \sigma_0^2(\partial_{xx}^2 u)^-] = 0, \ u(0,x) = \phi,$$

$$\sigma_0 \in [0,1] \quad \text{fixed}$$

nonlinear heat equation

$$\partial_t u - \frac{1}{2}[(\partial^2_{xx} u)^+ - \sigma_0^2 (\partial^2_{xx} u)^-] = 0, \; u(0,x) = \phi,$$

$$\sigma_0 \in [0,1] \quad \text{fixed}$$

or, G–heat equation

$$\partial_t u - G(\partial^2_{xx} u) = 0, \quad u(0,x) = \phi(x).$$

with

$$G(a) = \frac{1}{2}(a^+ - \sigma_0^2 a^-)$$

$$= \tfrac{1}{2} \sup_{\gamma \in [\sigma_0, 1]} \gamma a, \;\; a \in \mathbb{R}$$

Lemma. If ξ is standard G–normal distributed, then the function

$$u(t,x) = \mathbb{E}[\phi(x+\sqrt{t}\xi)]$$

is the solution of the G–heat equation with the initial condition $u(0,\cdot) = \phi(\cdot)$.

$$P_t^G(\phi)(x) = \mathbb{E}[\phi(x+\sqrt{t}\times\xi)],$$
$$(t,x) \in [0,\infty)\times\mathbb{R}.$$

Nonlinear Bachelier-Wiener-Kolmogorov–Chapman chain rule:

$$P_t^G(P_s^G(\phi))(x) = P_{t+s}^G(\phi)(x),$$

$$s,t \geq 0,\ x \in \mathbb{R}.$$

Proposition. For each $t > 0$, the G–normal distribution is a sublinear expectation on $lip(\mathbb{R})$, satisfying, for each $\Phi, \Psi \in lip(\mathbb{R})$,

(a) Monotonicity:

$$\Phi \geq \Psi \implies \mathbb{E}[\Phi(\xi)] \geq \mathbb{E}[\Psi(\xi)].$$

(b) Preserving of constants: $\mathbb{E}[c] = c$.

(c) Sub-additivity:

$$\mathbb{E}[\Phi(\xi)] - \mathbb{E}[\Psi(\xi)] \leq \mathbb{E}[\Phi(\xi) - \Psi(\xi)],$$

(d) Positive homogeneity:

$$\mathbb{E}[\lambda\Phi(\xi)] = \lambda\mathbb{E}[\Phi(\xi)],\ \forall\lambda \geq 0.$$

(e) Constant translatability:

$$\mathbb{E}[\Phi(\xi) + c] = \mathbb{E}[\Phi(\xi)] + c.$$

$\{\phi(\xi), \ \ \phi \in lip(\mathbb{R})\}$ forms a normed space:

$$\|\phi(\xi)\| := \mathbb{E}[|\phi(\xi)|]$$

with ξ^n, $\xi^n\psi(\xi)$ as special elements

For each convex ϕ we have

$\mathbb{E}[\phi(\sqrt{t}\xi)] = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} \phi(x) \exp(-\frac{x^2}{2t}) dx$

$\mathbb{E}[-\phi(\sqrt{t}\xi)] = \frac{-1}{\sqrt{2\pi t}\sigma_0} \int_{-\infty}^{\infty} \phi(x) \exp(-\frac{x^2}{2t\sigma_0^2}) dx$

In particular, we have

$$\mathbb{E}[\sqrt{t}\xi] = 0, \quad \mathbb{E}[\xi^{2n+1}] = \mathbb{E}[-\xi^{2n+1}]$$
$$\mathbb{E}[t\xi^2] = t, \quad \mathbb{E}[-t\xi^2] = -\sigma_0^2 t.$$

2 1–dimensional G–Brownian

$\Omega := C_0(\mathbb{R}^+)$

$(\omega_t)_{t\in\mathbb{R}^+}$, with $\omega_0 = 0$, $B_t(\omega) = \omega_t$

with the distance

$$\rho(\omega^1, \omega^2) := \sum_{i=1}^{\infty} 2^{-i}[(\max_{t\in[0,i]} |\omega_t^1 - \omega_t^2|) \wedge 1].$$

Consider the space of random variables:

$$L_{ip}^0(\mathcal{H}_T) := \{X(\omega) = \phi(B_{t_1}(\omega), \cdots, B_{t_m}(\omega)), \\ \phi \in lip(\mathbb{R}^m)\, t_1, \cdots, t_m \in [0, T], \forall m\}.$$

It is clear that $L_{ip}^0(\mathcal{H}_t) \subseteq L_{ip}^0(\mathcal{H}_T)$, for $t \leq T$. We also denote

$$L_{ip}^0(\mathcal{H}) := \bigcup_{n=1}^{\infty} L_{ip}^0(\mathcal{H}_n)$$

Def. Canonical process $B_t(\omega) = \omega_t$, $t \geq 0$,
is called a G–**Brownian motion** under the
G–expectation $\mathbb{E} : L^0_{ip}(\mathcal{H}) \longmapsto \mathbb{R}$
if B_t is G–normal distributed: the
G-expectation of $X = \phi(B_t)$ is:

$$\mathbb{E}[\phi(B_t)] = \mathbb{E}[\phi(\sqrt{t}\xi)], \ \phi \in lip(\mathbb{R})$$

If $X = \phi(B_T - B_t)$:

$$\mathbb{E}[\phi(B_T - B_t)] = \mathbb{E}[\phi(\sqrt{T-t}\xi)]$$

and $\forall \phi \in lip(\mathbb{R}^m)$, $0 \leq t_1 < \cdots < t_m < \infty$
we have

$$\mathbb{E}[\phi(B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_m} - B_{t_{m-1}})] = \phi_m$$

$$\phi \in lip(\mathbb{R}^m),\ 0 \le t_1 < \cdots < t_m < \infty$$

ϕ_m is obtained via the backward deduction:

$$\begin{aligned}
\phi_1(x_1, \cdots, x_{m-1}) &= \mathbb{E}[\phi(x_1, \cdots, x_{m-1}, \sqrt{t_m - t_{m-1}}\xi)] \\
\phi_2(x_1, \cdots, x_{m-2}) &= \mathbb{E}[\phi_1(x_1, \cdots, x_{m-2}, \sqrt{t_{m-1} - t_{m-2}}\xi)] \\
&\vdots \\
\phi_{m-1}(x_1) &= \mathbb{E}[\phi_{m-2}(x_1, \sqrt{t_2 - t_1}\xi)] \\
\phi_m &= \mathbb{E}[\phi_{m-1}(\sqrt{t_1}\xi)].
\end{aligned}$$

$\phi \in lip(\mathbb{R}^m)$, $0 \leq t_1 < \cdots < t_m < \infty$

The G–conditional expectation of $X = \phi(B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_m} - B_{t_{m-1}})$ under $\mathcal{H}_{t_{m-1}}$:

$$\mathbb{E}[X|H_{t_{m-1}}] := \mathbb{E}[\phi(B_{t_1}, \cdots, B_{t_m} - B_{t_{m-1}})|\mathcal{H}_{t_{m-1}}]$$
$$:= \phi_1(B_{t_1}, \cdots, B_{t_{m-1}} - B_{t_{m-2}}).$$

$\phi_1(x_1, \cdots, x_{m-1}) = \mathbb{E}[\phi(x_1, \cdots, x_{m-1}, B_{t_m} - B_{t_{m-1}})]$

Backward deduction:

$$\mathbb{E}[X|\mathcal{H}_{t_{m-2}}] \quad \mathbb{E}[X|\mathcal{H}_{t_{m-3}}] \quad \cdots \quad \mathbb{E}[X|\mathcal{H}_{t_1}] = \phi_m(B_{t_1})$$

The G–expectation of X: $\mathbb{E}[X] = \mathbb{E}[\phi_m(B_{t_1})]$

$\mathbb{E}[\cdot]$ consistently defines a sublinear expectation on $L^0_{ip}(\mathcal{H}_T)$ and $L^0_{ip}(\mathcal{H})$ satisfying (a)–(e).

Thus

$$\|X\| := \mathbb{E}[|X|],\ X \in L^0_{ip}(\mathcal{H}_T) \ \ (\text{resp. } \ L^0_{ip}(\mathcal{H}))$$

forms a norm

$L^0_{ip}(\mathcal{H}_T)$ (resp. $L^0_{ip}(\mathcal{H})$) can be continuously extended to a Banach space denoted by $L^1_G(\mathcal{H}_T)$ (resp. $L^1_G(\mathcal{H})$).

For each $0 \leq t \leq T < \infty$, we have

$L_G^1(\mathcal{H}_t) \subseteq L_G^1(\mathcal{H}_T) \subset L_G^1(\mathcal{H})$.

It is easy to check that, in $L_G^1(\mathcal{H}_T)$ (resp. $L_G^1(\mathcal{H}_T)$), $\mathbb{E}[\cdot]$ still satisfies (a)–(e).

Def. The expectation

$$\mathbb{E}[\cdot] : L_G^1(\mathcal{H}) \mapsto \mathbb{R}$$

is called G–expectation.

The canonical process B is called a G–Brownian motion under $\mathbb{E}[\cdot]$.

For each $p > 1$, we denote
$L_G^p(\mathcal{H}) = \{X \in L_G^1(\mathcal{H}),\ |X|^p \in L_G^1(\mathcal{H})\}$.
$L_G^p(\mathcal{H})$ is a Banach space under
$\|X\|_p := (\mathbb{E}[|X|^p])^{1/p}$. We have

$$\|X+Y\|_p \leq \|X\|_p + \|Y\|_p$$
$$\|XY\| = \mathbb{E}[|XY|] \leq \|X\|_p \|X\|_q,$$
$$\forall X \in L_G^p,\ Y \in L_G^q(Q),\ 1/p + 1/q = 1$$

Lemma. $\mathbb{E}[\cdot|\mathcal{H}_t] : L^0_{ip}(\mathcal{H}_T) \mapsto L^0_{ip}(\mathcal{H}_t)$ is a continuous mapping under $\|\cdot\|$ since:

$$\mathbb{E}[\mathbb{E}[X|\mathcal{H}_t] - \mathbb{E}[Y|\mathcal{H}_t]] \leq \mathbb{E}[X - Y],$$
$$\left\|\mathbb{E}[X|\mathcal{H}_t] - \mathbb{E}[Y|\mathcal{H}_t]\right\| \leq \|X - Y\|.$$

$\mathbb{E}[\cdot|\mathcal{H}_t]$ can be extended as a continuous mapping $L^1_G(\mathcal{H}_T) \mapsto L^1_G(\mathcal{H}_t)$.

Proposition. For each X, $Y \in L_G^1(\mathcal{H})$:

(i) $\mathbb{E}[X|\mathcal{H}_t] = X$, for $X \in L_G^1(\mathcal{H}_t)$, $t \leq T$.

(ii) If $X \geq Y$, then $\mathbb{E}[X|\mathcal{H}_t] \geq \mathbb{E}[Y|\mathcal{H}_t]$.

(iii) $\mathbb{E}[X|\mathcal{H}_t] - \mathbb{E}[Y|\mathcal{H}_t] \leq \mathbb{E}[X - Y|\mathcal{H}_t]$.

(iv)
$$\mathbb{E}[\mathbb{E}[X|\mathcal{H}_t]|\mathcal{H}_s] = \mathbb{E}[X|\mathcal{H}_{t\wedge s}],$$
$$\mathbb{E}[\mathbb{E}[X|\mathcal{H}_t]] = \mathbb{E}[X].$$

(v) $\mathbb{E}[X + \eta|\mathcal{H}_t] = \mathbb{E}[X|\mathcal{H}_t] + \eta$, $\eta \in L_G^1(\mathcal{H}_t)$.

(vi) $\mathbb{E}[\eta X|\mathcal{H}_t] = \eta^+\mathbb{E}[X|\mathcal{H}_t] + \eta^-\mathbb{E}[-X|\mathcal{H}_t]$,

$\forall\ \eta \in L_G^1(\mathcal{H}_t)$.

(vii) For each $X \in L_G^1(\mathcal{H}_T^t)$, $\mathbb{E}[X|\mathcal{H}_t] = \mathbb{E}[X]$.

where $L_G^1(\mathcal{H}_T^t)$ is the extension, under $\|\cdot\|$, of

$L_{ip}^0(\mathcal{H}_T^t) = \phi(B_{t_1} - B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_m} - B_{t_{m-1}}), \qquad m = 1, 2, \cdots,$
$\phi \in lip(\mathbb{R}^m)$, $t_1, \cdots, t_m \in [t, T]$.

Def. $X \in L_G^1(\mathcal{H})$ is said to be independent of $\mathcal{H}_t$ under $\mathbb{E}$ for some given $t \in [0, \infty)$, if for each real function Φ with $\Phi(X) \in L_G^1(\mathcal{H})$, we have

$$\mathbb{E}[\Phi(X)|\mathcal{H}_t] = \mathbb{E}[\Phi(X)].$$

Example. $\mathbb{E}[|B_t - B_s|^n|\mathcal{H}_s] = \mathbb{E}[|B_{t-s}|^{2n}]$

$$= \frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{\infty} |x|^n \exp(-\frac{x^2}{2(t-s)}) dx.$$

But

$$\mathbb{E}[-|B_t - B_s|^n|\mathcal{H}_s] = \mathbb{E}[-|B_{t-s}|^n] = -\sigma_0^n \mathbb{E}[|B_{t-s}|^n].$$

As in classical cases, we have

$$\mathbb{E}[(B_t - B_s)^2|\mathcal{H}_s] = t - s,$$

$$\mathbb{E}[(B_t - B_s)^4|\mathcal{H}_s] = 3(t-s)^2,$$

$$\mathbb{E}[(B_t - B_s)^6|\mathcal{H}_s] = 15(t-s)^3,$$

$$\mathbb{E}[(B_t - B_s)^8|\mathcal{H}_s] = 105(t-s)^4,$$

$$\mathbb{E}[|B_t - B_s||\mathcal{H}_s] = \frac{\sqrt{2(t-s)}}{\sqrt{\pi}}$$

$$\mathbb{E}[|B_t - B_s|^3|\mathcal{H}_s] = \frac{2\sqrt{2}(t-s)^{3/2}}{\sqrt{\pi}}$$

Example. For each $n=1,2,\cdots,\ 0\leq t<T$
and $X\in L_G^1(\mathcal{H}_t)$,
we have

$$\mathbb{E}[|X|(B_T-B_t)^{2n-1}]=\mathbb{E}[|X|]\cdot\mathbb{E}[B_{T-t}^{2n-1}]$$

$$\mathbb{E}[X(B_T-B_t)|\mathcal{H}_s]=\mathbb{E}[-X(B_T-B_t)|\mathcal{H}_s]=0$$

We also have

$$\mathbb{E}[X(B_T - B_t)^2|\mathcal{H}_t] = X^+(T-t) - \sigma_0^2 X^-(T-t).$$

$$E[X] - E[Y] \le \mathbb{E}[X-Y], \quad \forall X, Y \in L_{ip}^0(\mathcal{H})$$

Itô's integral of G–Brownian motion

Def. For $T \in \mathbb{R}_+$, a partition π_T of $[0,T]$ is a finite ordered subset $\pi_T^N = \{t_1, \cdots, t_N\}$ such that $0 = t_0 < t_1 < \cdots < t_N = T$.

$$\mu(\pi_T) = \max\{|t_{i+1} - t_i|, i = 0, 1, \cdots, N-1\}.$$

Def. For each $\eta \in M_G^{p,0}(0,T)$, i.e.,

$$\eta_t = \sum_{j=0}^{N-1} \xi_j(\omega)\mathbf{I}_{[t_j,t_{j+1})}(t),\ \xi_j \in L_G^p(\mathcal{H}_{t_j})$$

the related Bochner integral is

$$\int_0^T \eta_t(\omega)dt = \sum_{j=0}^{N-1} \xi_j(\omega)(t_{j+1} - t_j),\quad \xi_j \in L_G^1(\mathcal{H}_{t_j})$$

Def. For each $p \geq 1$, we will denote by $M_G^p(0,T)$ the completion of $M_G^{p,0}(0,T)$ under the norm

$$(\frac{1}{T}\int_0^T \left\|\eta_t^p\right\| dt)^{1/p}$$
$$= \left(\frac{1}{T}\sum_{j=0}^{N-1}\mathbb{E}[|\xi_j(\omega)|^p](t_{j+1}-t_j)\right)^{1/p}.$$

We observe that,

$$\mathbb{E}[|\int_0^T \eta_t(\omega)dt|] \leq \int_0^T \mathbb{E}[|\eta_t|]dt.$$

Proposition. The linear mapping $\int_0^T \eta_t(\omega)dt : M_G^{1,0}(0,T) \mapsto L_G^1(\mathcal{H}_T)$ is continuous. and thus can be continuously extended to $M_G^1(0,T) \mapsto L_G^1(\mathcal{H}_T)$. We still denote this extended mapping by $\int_0^T \eta_t(\omega)dt$, $\eta \in M_G^1(0,T)$. We have

$$\mathbb{E}[|\int_0^T \eta_t(\omega)dt|] \leq \int_0^T \mathbb{E}[|\eta_t|]dt, \quad \forall \eta \in M_G^1(0,T).$$

Itô's integral of G–Brownian motion

Def. For each $\eta \in M_G^{2,0}(0,T)$ with the form

$$\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) \mathbf{I}_{[t_j, t_{j+1})}(t),$$

we define

$$I(\eta) = \int_0^T \eta(s) dB_s := \sum_{j=0}^{N-1} \xi_j (B_{t_{j+1}} - B_{t_j}).$$

Lemma. $I : M_G^{2,0}(0,T) \longmapsto L_G^2(\mathcal{H}_T)$ is a linear continuous mapping and thus can be continuously extended to

$I : M_G^2(0,T) \longmapsto L_G^2(\mathcal{H}_T)$:

$$\mathbb{E}[\int_0^T \eta(s)dB_s] = 0,$$

$$\mathbb{E}[(\int_0^T \eta(s)dB_s)^2] \leq \int_0^T \mathbb{E}[(\eta(t))^2]dt.$$

Def. We define, the stochastic integral

$$\int_0^T \eta(s)dB_s := I(\eta), \quad \eta \in M_G^2(0,T)$$

$$\int_s^t \eta_u dB_u := \int_0^T \mathbf{I}_{[s,t]}(u)\eta_u dB_u,$$

Proposition. We have

(i) $\int_s^t \eta_u dB_u = \int_s^r \eta_u dB_u + \int_r^t \eta_u dB_u$.

(ii) $\int_s^t (\alpha\eta_u + \theta_u) dB_u = \alpha \int_s^t \eta_u dB_u + \int_s^t \theta_u dB_u$

$$\alpha \in L_G^1(\mathcal{H}_s)$$

(iii) $\mathbb{E}[X + \int_r^T \eta_u dB_u | \mathcal{H}_s] = \mathbb{E}[X]$, $\forall X \in L_G^1(\mathcal{H})$.

Quadratic variation process of G–BM

π_t^N, $N = 1, 2, \cdots$: partitions of $[0, t]$.

$$B_t^2 = \sum_{j=0}^{N-1} [B^2_{t^N_{j+1}} - B^2_{t^N_j}]$$
$$= \sum_{j=0}^{N-1} 2B_{t^N_j}(B_{t^N_{j+1}} - B_{t^N_j}) + \sum_{j=0}^{N-1} (B_{t^N_{j+1}} - B_{t^N_j})^2$$

the 1st term tends to $\int_0^t B_s dB_s$.

We denote:

$$\langle B\rangle_t = \lim_{\mu(\pi_t^N)\to 0} \sum_{j=0}^{N-1} (B_{t_{j+1}^N} - B_{t_j^N})^2 = B_t^2 - 2\int_0^t B_s dB_s.$$

$\langle B\rangle_t$, $t \geq 0$, is an increasing process called **quadratic variation process** of B.

$\langle B \rangle_t$ is not a deterministic process unless

$\sigma = 1$!!!

Lemma. $\mathbb{E}[\langle B \rangle_{s+t} - \langle B \rangle_s | \mathcal{H}_s] = t,$

$$\mathbb{E}[-(\langle B \rangle_{s+t} - \langle B \rangle_s) | \mathcal{H}_s] = -\sigma_0^2 t,$$

The integration w.r.t. $d\langle B\rangle$:

$\eta_t(\omega)=\sum_{j=0}^{N-1}\xi_j(\omega)\mathbf{I}_{[t_j,t_{j+1})}(t)\in M_G^{1,0}(0,T)$

$$Q_{0,T}(\eta)=\int_0^T\eta(s)d\langle B\rangle_s:=\sum_{j=0}^{N-1}\xi_j(\langle B\rangle_{t_{j+1}}-\langle B\rangle_{t_j})$$

$$Q_{0,T}(\eta):M_G^{1,0}(0,T)\mapsto L^1(\mathcal{H}_T).$$

Lemma. For each $\eta \in M_G^{1,0}(0,T)$,

$$\mathbb{E}[|\int_0^T \eta(s) d\langle B \rangle_s|] \leq \int_0^T \mathbb{E}[|\eta_s|] ds,$$

Thus $Q_{0,T} : M_G^{1,0}(0,T) \mapsto L^1(\mathcal{H}_T)$ is a continuous linear mapping and

$$\mathbb{E}[|\int_0^T \eta(s) d\langle B \rangle_s|] \leq \int_0^T \mathbb{E}[|\eta_s|] ds, \quad \forall \eta \in M_G^1(0,T).$$

Lemma. For each fixed $s \geq 0$, $(\langle B\rangle_{s+t} - \langle B\rangle_s)_{t\geq 0}$ is independent of $\mathcal{H}_s$. It is the quadratic variation process of the Brownian motion $B_t^s = B_{s+t} - B_s$, $t \geq 0$, i.e., $\langle B\rangle_{s+t} - \langle B\rangle_s = \langle B^s\rangle_t$.

We have

$$\mathbb{E}[\langle B^s\rangle_t^2 | \mathcal{H}_s] = \mathbb{E}[\langle B\rangle_t^2] = t^2.$$

as well as

$$\mathbb{E}[\langle B^s\rangle_t^3 | \mathcal{H}_s] = \mathbb{E}[\langle B\rangle_t^2] = t^3,$$
$$\mathbb{E}[\langle B^s\rangle_t^4 | \mathcal{H}_s] = \mathbb{E}[\langle B\rangle_t^4] = t^4.$$

Proposition. $\forall\ 0 \le s \le t$, $\xi \in L_G^1(\mathcal{H}_s)$,

$$\mathbb{E}[X + \xi(B_t^2 - B_s^2)] = \mathbb{E}[X + \xi(\langle B \rangle_t - \langle B \rangle_s)].$$

Proposition. We have the following isometry

$$\mathbb{E}[(\int_0^T \eta(s)dB_s)^2] = \mathbb{E}[\int_0^T \eta^2(s)d\langle B\rangle_s],$$

$$\eta \in M_G^2(0,T)$$

Itô's formula for G–Brownian motion

Consider

$$X_t^\nu = X_0^\nu + \int_0^t \alpha_s^\nu ds + \int_0^t \eta_s^\nu d\langle B\rangle_s + \int_0^t \beta_s^\nu dB_s$$

Theorem Let α^ν, β^ν and η^ν, $\nu = 1, \cdots, n$, are bounded processes of $M_G^2(0,T)$. Then for each $t \geq 0$ and in $L_G^2(\mathcal{H}_t)$ we have

$$\begin{aligned}\Phi(X_t) = \Phi(X_s) &+ \int_s^t \partial_{x^\nu}\Phi(X_u)\beta_u^\nu dB_u \\ &+ \int_s^t \partial_{x_\nu}\Phi(X_u)\alpha_u^\nu du \\ &+ \int_s^t [\partial_{x^\nu}\Phi(X_u)\eta_u^\nu + \tfrac{1}{2}\partial^2_{x^\mu x^\nu}\Phi(X_u)\beta_u^\mu\beta_u^\nu] d\langle B\rangle_u\end{aligned}$$

3 Stochastic differential equations

We consider the following SDE: $X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t h(X_s)d\langle B\rangle_s + \int_0^t \sigma(X_s)dB_s,\ t \in [0,T]$.where $X_0 \in \mathbb{R}^n$ is given and

$b, h, \sigma : \mathbb{R}^n \mapsto \mathbb{R}^n$ are given Lip. functions.

The solution: a process $X \in M_G^2(0,T;\mathbb{R}^n)$

satisfying the above SDE.

We introduce a mapping:

$$\Lambda_{\cdot}(Y) :=: Y \in M_G^2(0,T;\mathbb{R}^n) \longmapsto M_G^2(0,T;\mathbb{R}^n)$$

by

$$\Lambda_t(Y) = X_0 + \int_0^t b(Y_s)ds + \int_0^t h(Y_s)d\langle B\rangle_s + \int_0^t \sigma(Y_s)dB_s,\ t \in [0,T].$$

Lemma. For each $Y, Y' \in M_G^2(0,T;\mathbb{R}^n)$, we have the following estimate:

$$\mathbb{E}[|\Lambda_t(Y) - \Lambda_t(Y')|^2] \leq C\int_0^t \mathbb{E}[|Y_s - Y_s'|^2]ds,$$

where $C = 3K^2$.

Theorem. There exists a unique solution $X \in M_G^2(0,T;\mathbb{R}^n)$ of the stochastic differential equation.

G–expectation $\mathbb{E}[\cdot]$ cannot be in a framework of $(\Omega, \mathcal{F}, P)$

But the classical Wiener expectation $E[\cdot]$ is dominated by $\mathbb{E}[\cdot]$:

$$E[X] - E[Y] \leq \mathbb{E}[X - Y], \quad \forall X, Y \in L_G^1(\mathcal{H}).$$

Particularly $(B_t)_{t \geq 0}$ is a classical BM under E.

$E[|X|]$, $X \in L_G^1(\mathcal{H})$ is the classical L^1–norm and

$$E[|\langle B \rangle_t - t|] = 0, \quad \forall t \geq 0.$$

In particular the above Itô's formula becomes the classical one under L^1–norm

If $\bar{G}_1(a) = \frac{1}{2}(\gamma a^+ - \gamma_0 a^-)$, $0 \leq \gamma_0 \leq \sigma_0 \leq \gamma$

Then we can construct $L^1_{G_1}(\mathcal{H})$ space and related G_1–expectation $\mathbb{E}_1$ on $L^1_{G_1}(\mathcal{H})$, with

$$\mathbb{E}[X] - \mathbb{E}[Y] \leq \mathbb{E}_1[X - Y], \quad X, Y \in L^1_{G_1}(\mathcal{H}).$$

$\mathbb{E}_1[\cdot]$ forms a risk measure,

$\mathbb{E}[\cdot]$ is well-defined on $L^1_{G_1}(\mathcal{H})$ but not inversely

$\mathbb{E}_1[\cdot]$ dominates many dynamic risk measures

Example. Stock price $(S_t)_{t\geq 0}$ on $L^1_G(\mathcal{H})$

$$dS_t = S_t(\mu dt + \overline{\mu} d\langle B\rangle_t + \overline{\sigma} dB_t), \quad S_0 = s.$$

Different $\mathbb{E}_0$ dominated by $\mathbb{E}$ "reads" this formula differently under the same framework of $L^1_G(\mathcal{H})$.

For example a linear $\mathbb{E}_0$ can "read" B_t as

$$B_t = \int_0^t \phi_s dW_s$$

where W is a $\mathbb{E}_0$-classsical BM and

$$\sigma_0 \leq \phi_s \leq 1, \quad \text{a.s. under } \mathbb{E}_0[\cdot]$$

There is no ambiguity in the pointview of $\mathbb{E}_0[\cdot]$.

We have $\langle B\rangle_t = \int_0^t |\phi_s|^2 ds$ under $\mathbb{E}_0[\cdot]$.