Generalised Deviations are Counterparts to Risk Measures

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joint presentation with

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Outline

- **Risk Management: Quick Introduction**
- **General Deviation Measures**
- **Risk versus Deviation**
- **Portfolio Optimization**
- **Capital Asset Pricing Model (CAPM)**
- **Generalized Linear Regression**

Outline

- **Risk Management: Quick Introduction**

  Several facts presented in this section are based on papers:

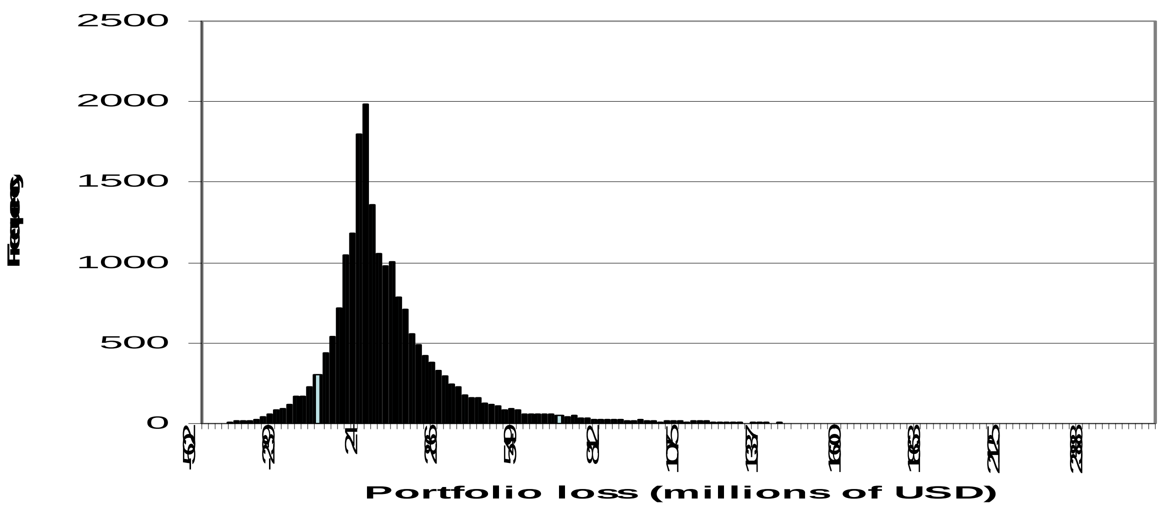
  - Rockafellar R.T. and S. Uryasev. Conditional Value-at-Risk for General Loss Distributions. Journal of Banking and Finance, 2002, 26/7 (www.ise.ufl.edu/uryasev/cvar2_jbf.pdf)

  - Rockafellar R.T. and S. Uryasev. Optimization of Conditional Value-at-Risk. The Journal of Risk. Vol. 2, No. 3, 2000, 21-41 (www.ise.ufl.edu/uryasev/cvar.pdf)

- **General Deviation Measures**
- **Risk versus Deviation**
- **Portfolio Optimization**
- **Capital Asset Pricing Model (CAPM)**
- **Generalized Linear Regression**

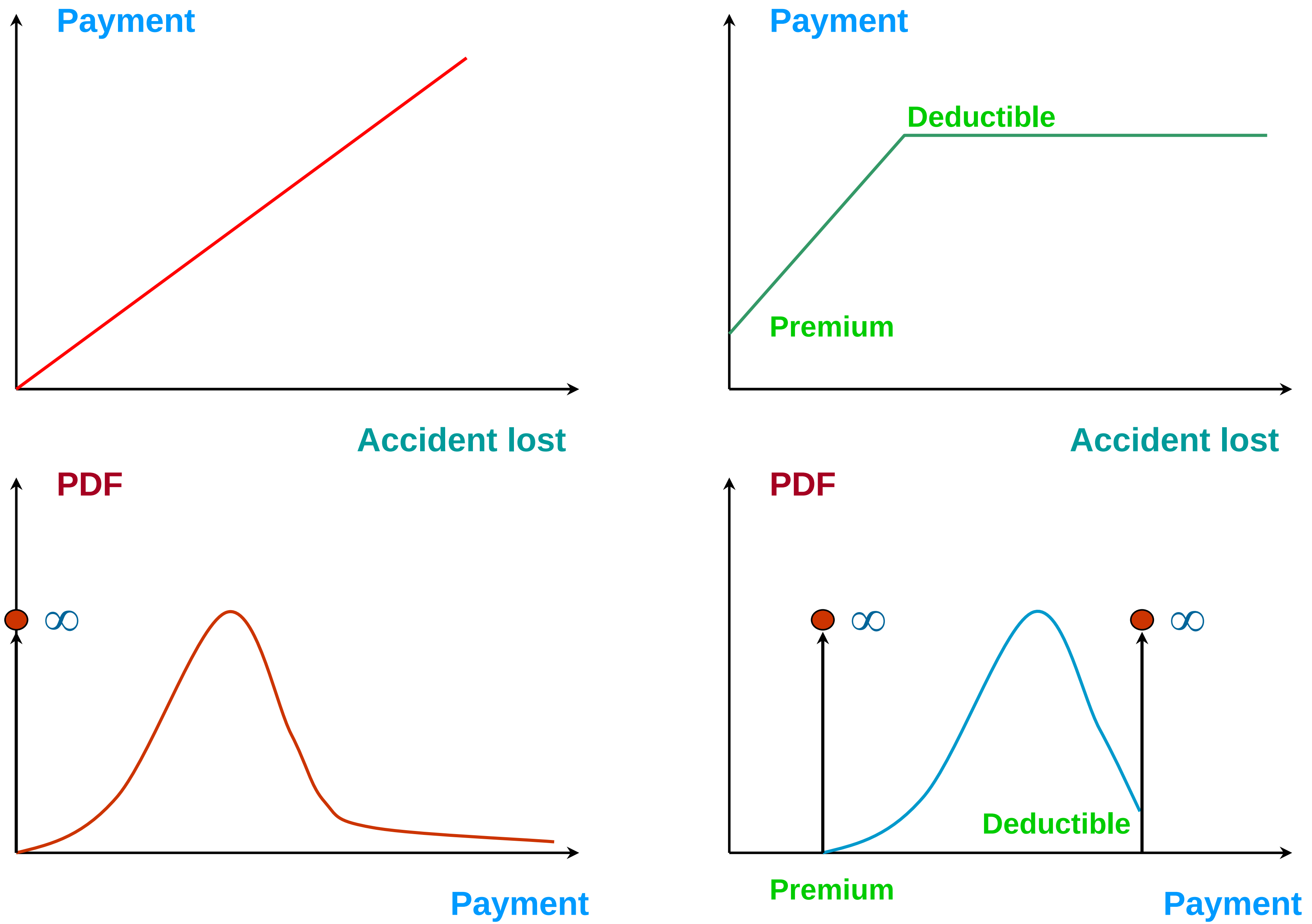
RISK MANAGEMENT

System with uncertain outcomes



<u>Risk Management</u> is a set of actions preventing undesirable outcomes and reshaping the loss distribution

RISK MANAGEMENT: INSURANCE



TWO CONCEPTS OF RISK

- **Risk as a possible loss**

  Minimum amount of cash to be added to make a portfolio (or project) sufficiently safe

  Example 1. MaxLoss

  - Three equally probable outcomes, { -4, 2, 5 }; MaxLoss = -4; Risk = 4
  - Three equally probable outcomes, { 0, 6, 9 }; MaxLoss = 0; Risk = 0

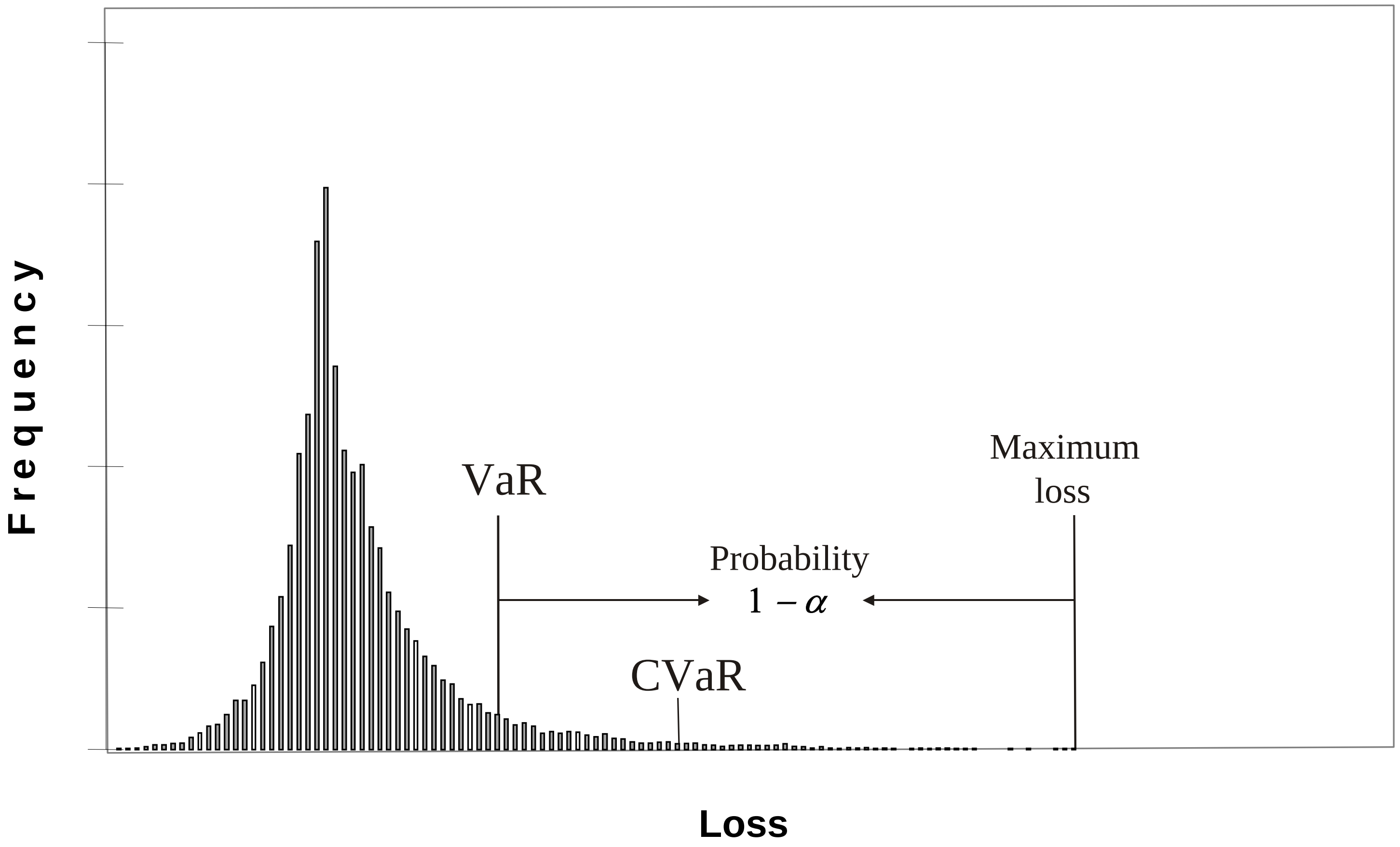
- **Risk as an uncertainty in outcomes**

  Some measure of deviation in outcomes

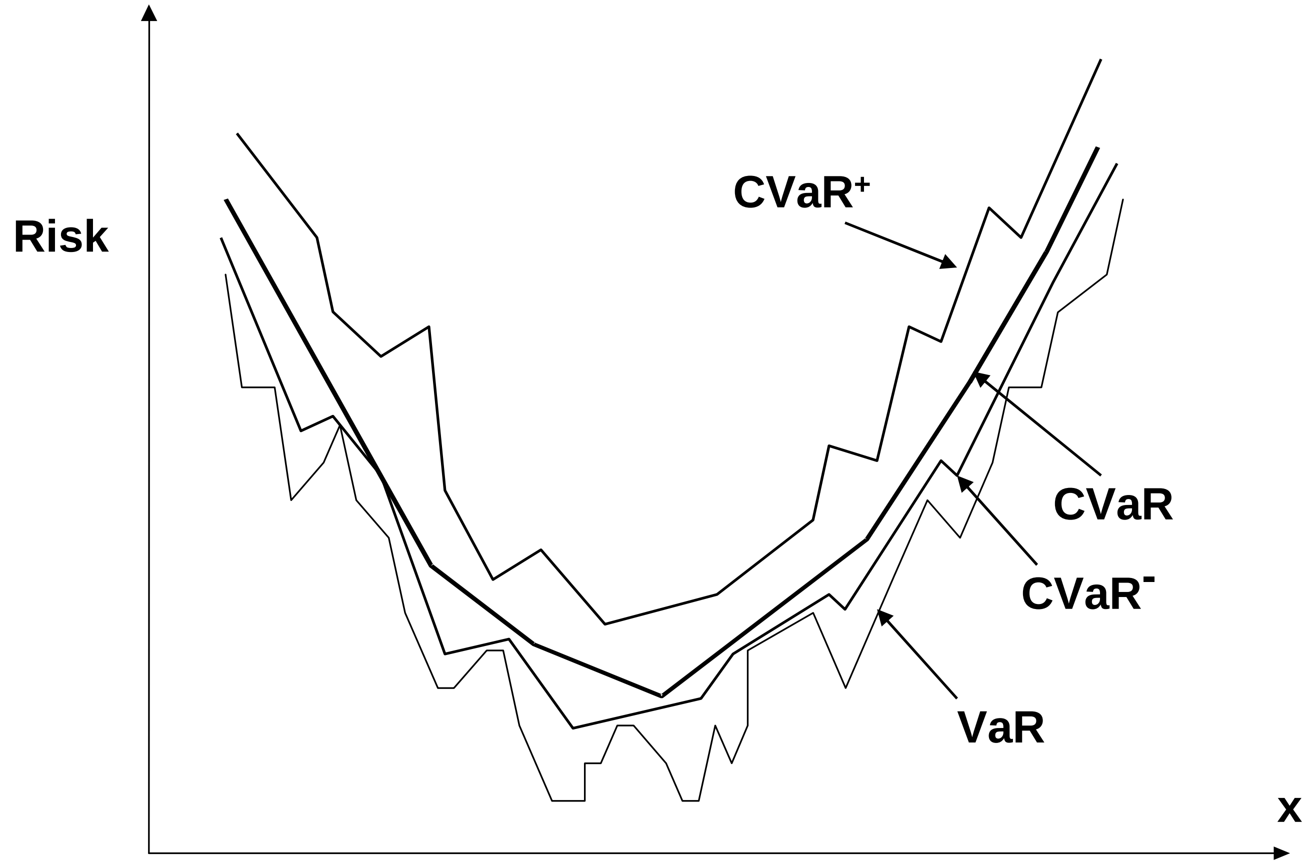
  Example 2. Standard Deviation

  - Three equally probable outcomes, { 0, 6, 9 }; Standard Deviation > 0

VaR, CVaR, DEVIATIONS



VaR, CVaR, $CVaR^+$ and $CVaR^-$



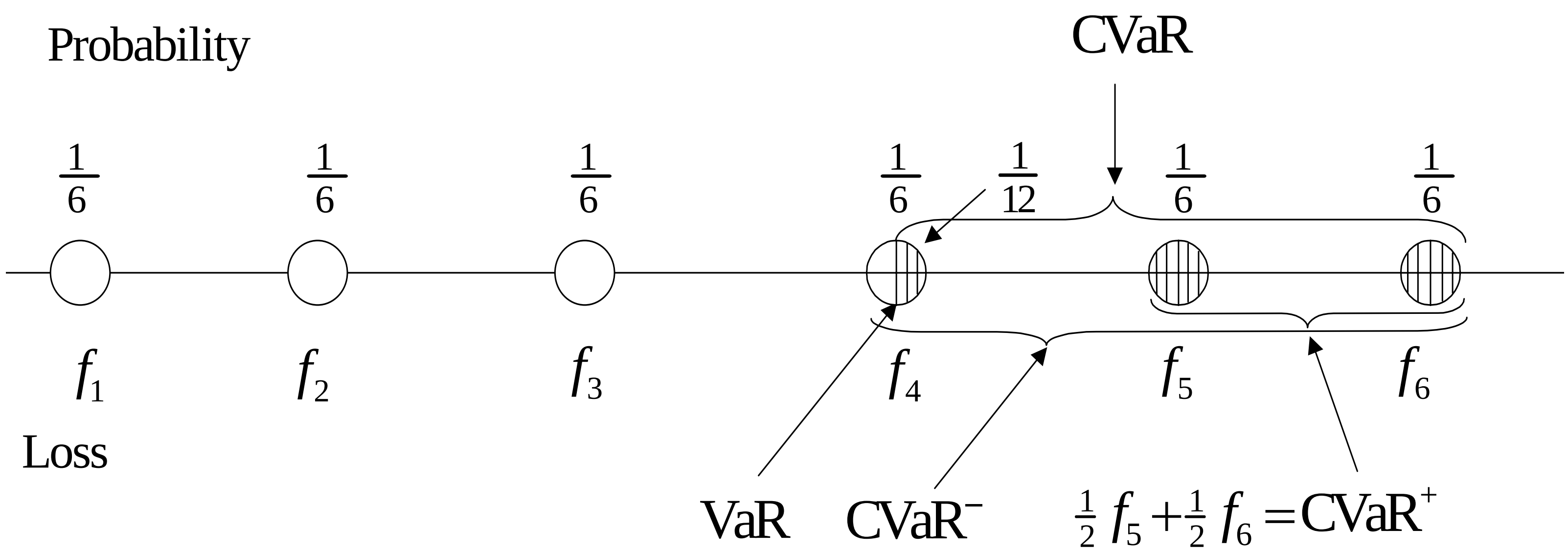
CVaR is convex, but VaR, $CVaR^-$, $CVaR^+$ may be non-convex, inequalities are valid: VaR $\leq$ $CVaR^-$ $\leq$ CVaR $\leq$ $CVaR^+$

CVaR: DISCRETE DISTRIBUTION, EXAMPLE

- α "splits" the atom:

$$\text{Six scenarios,}\quad p_1 = p_2 = \cdots = p_6 = \tfrac{1}{6},\ \alpha = \tfrac{7}{12}$$

$$\text{CVaR} = \tfrac{1}{5}\text{VaR} + \tfrac{4}{5}\text{CVaR}^+ = \tfrac{1}{5} f_4 + \tfrac{2}{5} f_5 + \tfrac{2}{5} f_6$$



Outline

- **Risk Management: Quick Introduction**
- **General Deviation Measures**

  Artzner, P., Delbaen F., Eber, J. M., and Heath, D.
  Coherent measures of risk. Mathematical Finance, 9, 1999, 203-228.

  Rockafellar, R. T., Uryasev, S. and M. Zabarankin:

  - **Generalized Deviations in Risk Analysis**. Finance and Stochastics, 10, 2006, 51-74
  - **Master Funds in Portfolio Analysis with General Deviation Measures**, The Journal of Banking and Finance, Vol. 30, #2, 2006

- **Risk versus Deviation**
- **Portfolio Optimization**
- **Capital Asset Pricing Model (CAPM)**
- **Generalized Linear Regression**

Standard Deviation

$$\sigma(X) = \sqrt{E[X - EX]^2}$$

Properties

- (D1) – insensitivity to constant shift

  $\sigma(X + C) = \sigma(X)$ *for all* X *and constants* C

- (D2) – positive homogeneity

  $\sigma(\lambda X) = \lambda\sigma(X)$ *for all* X *and all* $\lambda > 0$

- (D3) – subadditivity

  $\sigma(X + X') \leq \sigma(X) + \sigma(X')$ *for all* X *and* X'

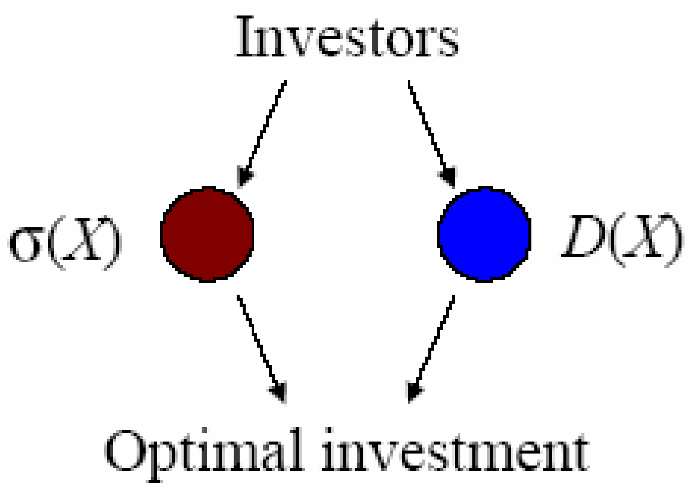
- (D4) – nonnegativity

  $\sigma(X) \geq 0$ *(equality for constant* X*)*

Introduction to Deviation Measures

Motivation

- standard deviation
- risk preferences
- nonsymmetric measures
- axiomatic approach



Previous work

- Artzner, Delbaen, Eber, and Heath, *Coherent Measures of Risk*

Motivation

Reproduce basic financial theory:

- Portfolio optimization
- CAPM
- Master fund
- Sharpe Ratio
- Two fund theorem

Deviation:

Asset beta (CAPM) $$\beta_i = \frac{\text{cov}(r_M, r_i)}{\sigma^2(r_M)}$$

Sharpe ratio $$S_i = \frac{E[r_i] - r_f}{\sigma(r_i)}$$

Portfolio optimization $$\min \sigma\left(x_1 r_1 + \ldots + x_n r_n\right)$$
$$s.t. \quad x_1 E r_1 + \ldots + x_n E r_n \geq \overline{r}$$

Risk: Associated with losses

More capital - less risk

Introduction to Deviation Measures (cont'd)

Theory of general deviation measures

- **unifying system of axioms**
- **continuous and discrete distributions**
- **non-differentiability**
- **broad range of applications**

Rockafellar, Uryasev, and Zabarankin

- *Deviation Measures in Risk Analysis and Optimization*
- *Portfolio Analysis with General Deviation Measures*
- *Deviation Measures in Generalized Linear Regression*

Deviation Measures

System of Axioms

- **(D1) – insensitivity to constant shift**
  $\mathcal{D}(X+C)=\mathcal{D}(X)$ *for all* X *and constants* C
- **(D2) – positive homogeneity**
  $\mathcal{D}(\lambda X)=\lambda\mathcal{D}(X)$ *for all* X *and all* $\lambda>0$
- **(D3) – subadditivity**
  $\mathcal{D}(X+X')\leq\mathcal{D}(X)+\mathcal{D}(X')$ *for all* X *and* X'
- **(D4) – nonnegativity**
  $\mathcal{D}(X)\geq 0$ *(equality for constant* X*)*

Deviation measure

- **(D5) – coherency**
  $\mathcal{D}(X)\leq EX-\inf X$ *for all* X

Coherent deviation measure

Examples of Deviation Measures

- **Standard Deviation**

$$\sigma(X) = \left(E[X - EX]^2\right)^{\frac{1}{2}}$$

- **Standard Semideviations**

$$\sigma_+(X) = (E[\max\{X - EX, 0\}^2])^{1/2}$$
$$\sigma_-(X) = (E[\max\{EX - X, 0\}^2])^{1/2}$$

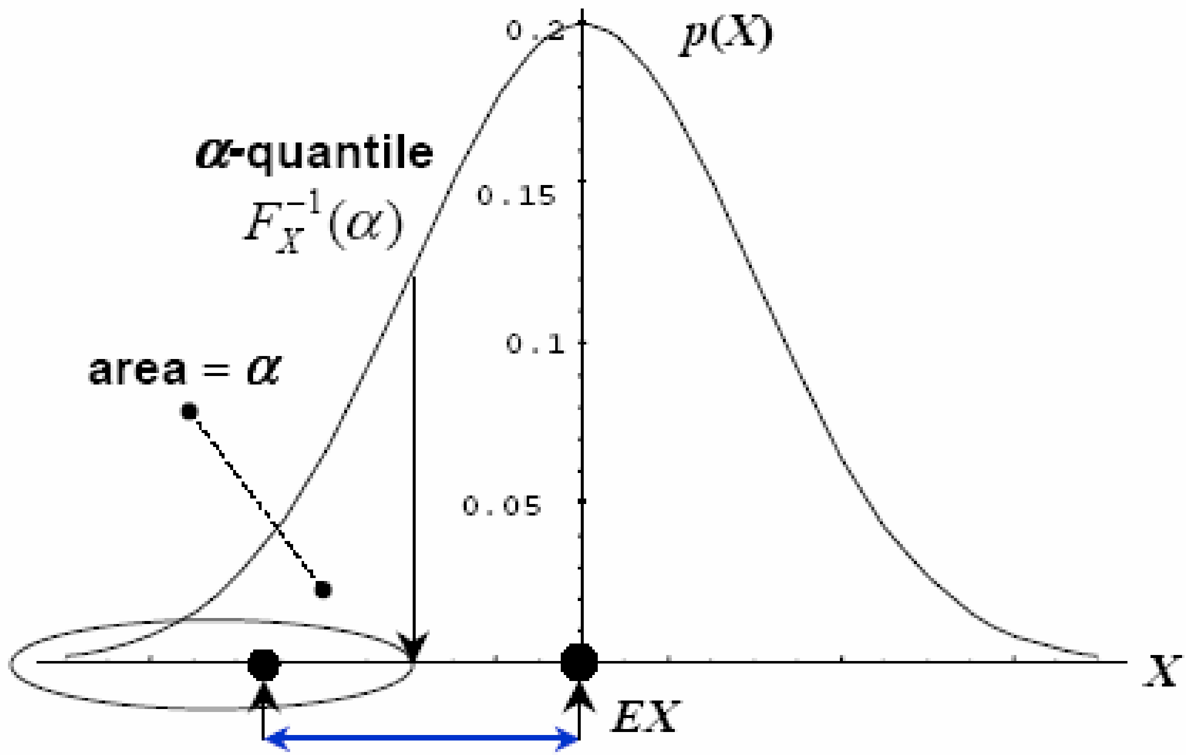
- **Deviation measure from range**

$$EX - \inf X$$

- **Generalized Mean Absolute Deviation (GMAD) with** $a(\omega) > 0$

$$\text{MAD}(X) = \int_\Omega a(\omega)|X(\omega) - EX|\,dP(\omega)$$

Deviation CVaR



Deviation Conditional Value-at-Risk (CVaR) for $\alpha \in [0,1]$

$$\mathrm{CVaR}_{\alpha}^{\Delta}(X) = EX - \frac{1}{\alpha}\int_{0}^{\alpha} F_X^{-1}(p)\,dp$$

Mixed Deviation CVaR

Mixed Deviation CVaR

- $\lambda_k \geq 0$ and $\sum_{k=1}^{K} \lambda_k = 1$

$$Mixed\text{-}\mathrm{CVaR}_{\alpha}^{\Delta}(X) = \sum_{k=1}^{K} \lambda_k \mathrm{CVaR}_{\alpha_k}^{\Delta}(X)$$

- $\lambda(\alpha) > 0$ and $\int_0^1 \lambda(\alpha)\, d\alpha = 1$

$$Mixed\text{-}\mathrm{CVaR}_{\alpha}^{\Delta}(X) = \int_0^1 \mathrm{CVaR}_{\alpha}^{\Delta}(X) d\lambda(\alpha)$$

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- **Portfolio Optimization**
- **Capital Asset Pricing Model (CAPM)**
- **Generalized Linear Regression**

Risk Measures

Expectation-Bounded Risk Measures

- (R1) – **constant translation**
  $\mathcal{R}(X + C) = \mathcal{R}(X) - C$ for all X and constants C

- (R2) – **positive homogeneity**
  $\mathcal{R}(0) = 0$, and $\mathcal{R}(\lambda X) = \lambda \mathcal{R}(X)$ for all X and all $\lambda > 0$

- (R3) – **subadditivity**
  $\mathcal{R}(X + X') \leq \mathcal{R}(X) + \mathcal{R}(X')$ for all X and X'

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- (R4) – **expectation-boundedness**
  $\mathcal{R}(X) \geq E[-X]$ (equality for constant X)

Risk measure

Artzner, Delbaen, Eber, and Heath

- (R5) – **coherency**
  $\mathcal{R}(X) \leq \mathcal{R}(X')$ when $X \geq X'$ (almost surely)

Coherent risk measure

Deviation versus Risk

Theorem

Deviation measures correspond one-to-one with **expectation-bounded** risk measures

(a) $\mathcal{D}(X) = \mathcal{R}(X - EX)$

(b) $\mathcal{R}(X) = \mathcal{D}(X) - EX$

- $\mathcal{R}$ **is coherent** $\iff$ $\mathcal{D}$ **is coherent**

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  Rockafellar, R. T., Uryasev, S. and M. Zabarankin:
  Master Funds in Portfolio Analysis with General Deviation Measures,
  The Journal of Banking and Finance, Vol. 30, #2, 2006

- **Capital Asset Pricing Model (CAPM)**
- **Generalized Linear Regression**

Notations and Assumptions

- **Market**
  - risk-free asset with constant rate r_0
  - n risky assets with rates $r = (r_1, \ldots, r_n), \quad r_j \in \mathcal{L}^2$
- **Portfolio**
  - weights: $x_0, \; x = (x_1, \ldots, x_n)$
  - budget constraint: $x_0 + x^\top e = 1$
  - rate of return: $X_p = x_0 r_0 + x^\top r$
- **No Redundancy Assumption**

  Only 0-portfolio, has risk-free return

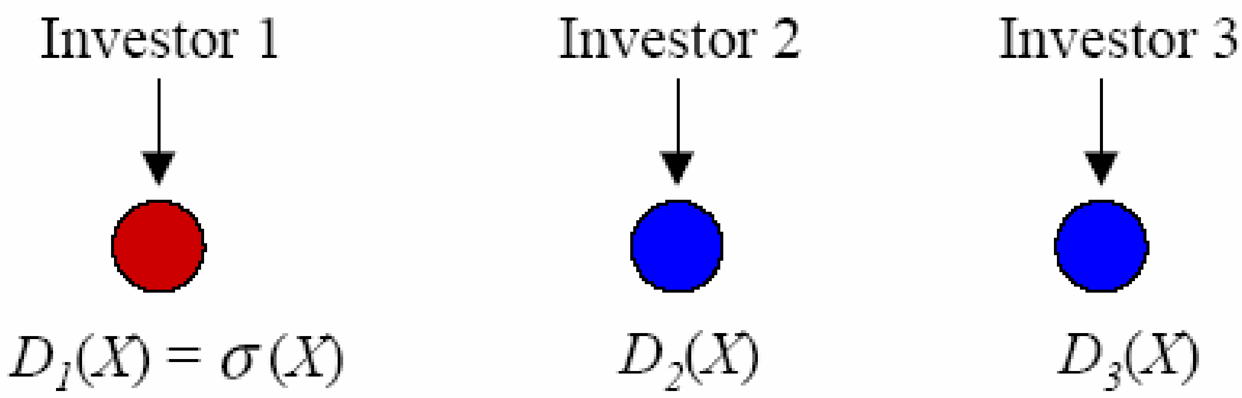
$$\mathcal{D}(x^\top r) > 0 \text{ for all } x \neq 0$$

- **No arbitrage Assumption**

  No x-portfolio has a *risk-free* return $> r_0$

Portfolio Optimization

Risk preferences

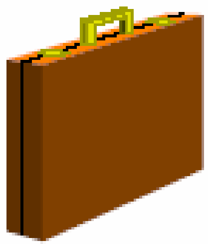


Problem formulation

$$d_0(\Delta) = \min_{x_0,\, x} \; \mathcal{D}(X_p) \quad \text{s. t.} \quad EX_p = r_0 + \Delta, \quad x_0 + x^\top e = 1$$

Δ = demanded additional gain over the risk-free rate

Portfolio Decomposition



Portfolio $(x_0, x_1, \ldots, x_n)$

$X_p = x_0 r_0 + (1-x_0) Y_p$

Risk free rate

r_0

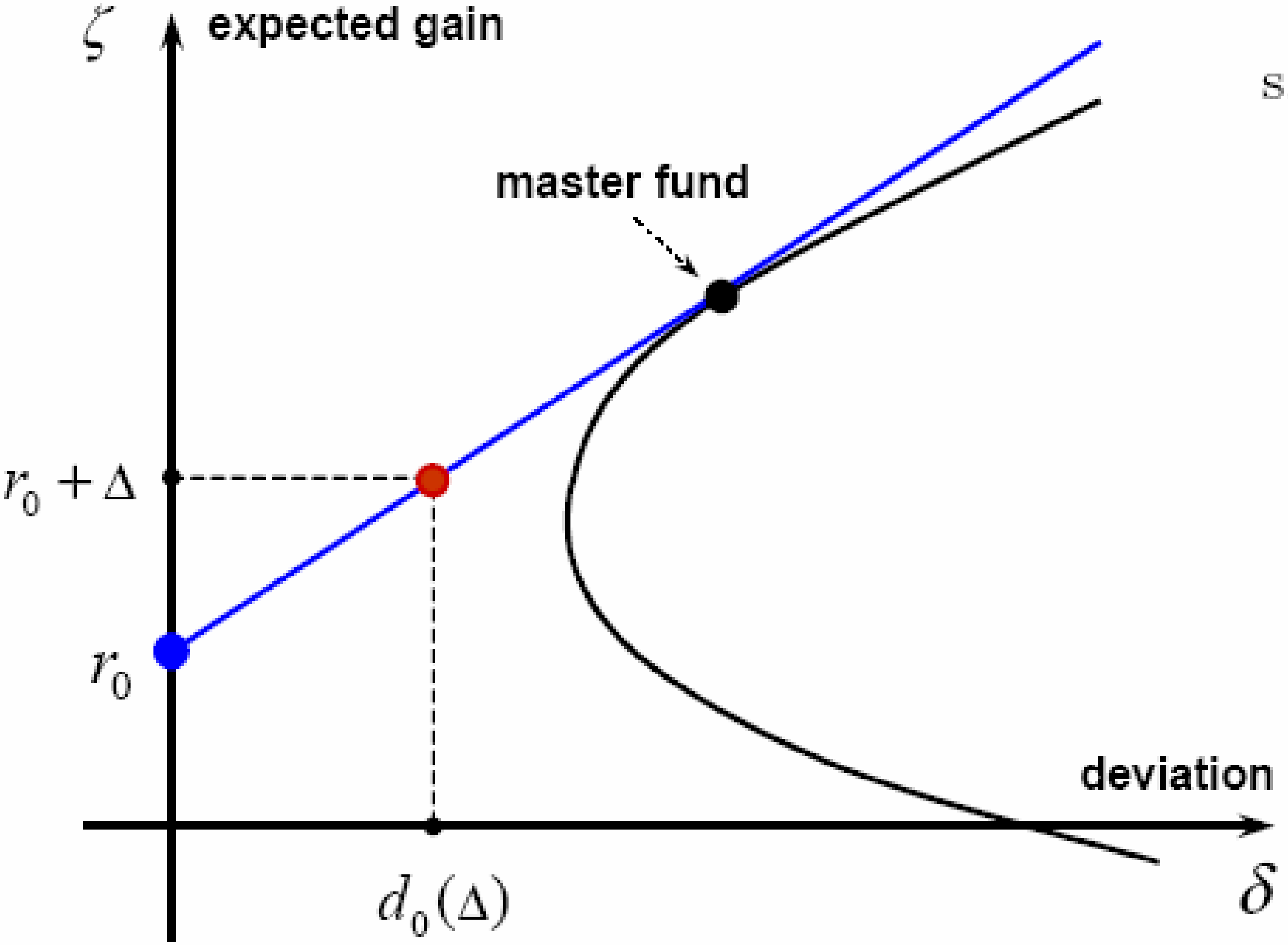
Portfolio of risky assets

$Y_p = r_0 y_1 + \ldots + r_n y_n$

- low r_0 ⟶ $y_1 + \ldots + y_n = 1$
- high r_0 ⟶ $y_1 + \ldots + y_n = -1$
- some r_0 ⟶ $y_1 + \ldots + y_n = 0$

Efficient Set: Classical Theory

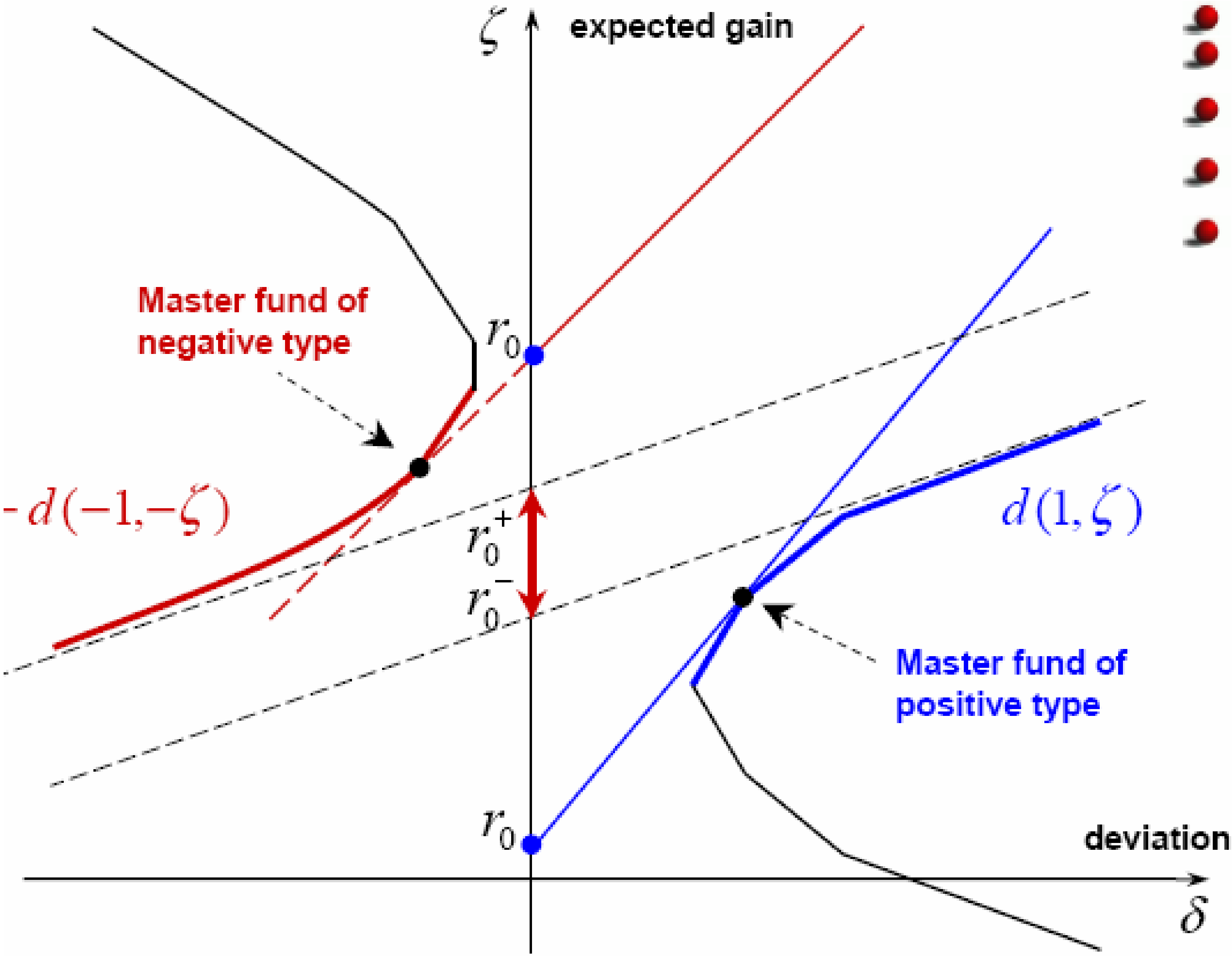
One Fund Theorem



Auxiliary Problem

$$\begin{aligned} \min_{y} \quad & \sigma(Y_p) \\ \text{s. t.} \quad & EY_p = \zeta \\ & y^\top e = 1 \end{aligned}$$

Efficient Sets: General Deviation



- convex
- slopes
- non-smooth
- non-symmetric
- threshold interval

Master Funds

Generalized One Fund Theorem

Existence of Master Funds

The threshold values $\hat{r}_0^+$ and $\hat{r}_0^-$ have the property that

- $r_0 < \hat{r}_0^- \implies$ master fund of positive type but none of negative type
- $r_0 > \hat{r}_0^+ \implies$ master fund of negative type but none of positive type
- $\hat{r}_0^- < r_0 < \hat{r}_0^+ \implies$ neither master fund of positive type nor one of negative type

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  Rockafellar, R. T., Uryasev, S. and M. Zabarankin:

  - Optimality Conditions in Portfolio Analysis with Generalized Deviation Measures, Mathematical Programming, accepted for publication, 2006

  - Equilibrium with Investors Using a Diversity of Deviation Measures. The Journal of Banking and Finance, accepted for publication, 2006

- **Generalized Linear Regression**

Duality and Risk Identifiers

Dual representation of deviation measures:

$$\mathcal{D}(X) = EX - \inf_{Q \in \mathcal{Q}} E[QX] = \sup_{Q \in \mathcal{Q}} \left(EX - E[QX] \right)$$

Risk envelope $\mathcal{Q}$ is dual to the deviation measure $\mathcal{D}$.

Risk identifier $\mathcal{Q}^*(X)$ is a member of risk envelope on which the deviation measure $\mathcal{D}(X)$ is attained:

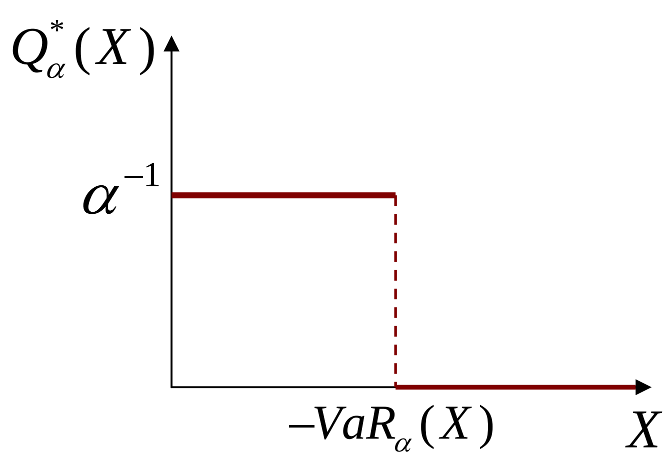
$$Q^*(X) = \arg\max_{Q \in \mathcal{Q}} \left(EX - E[QX] \right)$$

Example: CVaR-deviation

$$\mathcal{D}(X) = CVaR_\alpha(X - EX)$$

$$\mathcal{Q}_\alpha = \left\{ Q \,\middle|\, EQ = 1, 0 \le X \le \alpha^{-1} \right\}$$

For continuous random variable *X*, the risk identifier $Q^*_\alpha(X)$ is a step function.



Optimality Conditions and CAPM

Assumption: low risk-free rate, $r_0 < r_0^-$

- **Capital Asset Pricing Model (CAPM)**

$$\bar{r}_i - r_0 = \beta_i(\bar{r}_M - r_0), \qquad \beta_i = \frac{\mathrm{covar}(G, r_i)}{\mathcal{D}(r_M)}$$

G = risk identifier for the market fund r_M

CAPM Relations as Pricing Formula

General equation at the previous page implies:

Pricing formulas for an asset with price π_i and future payoff ς_i .

Pricing form of the CAPM-like relations:

$$\pi_i = \frac{E\varsigma_i}{1 + r_0 - \frac{Er_M - r_0}{\mathcal{D}(r_M)} \operatorname{cov}(r_i, G)}$$

Certainty equivalent pricing formula:

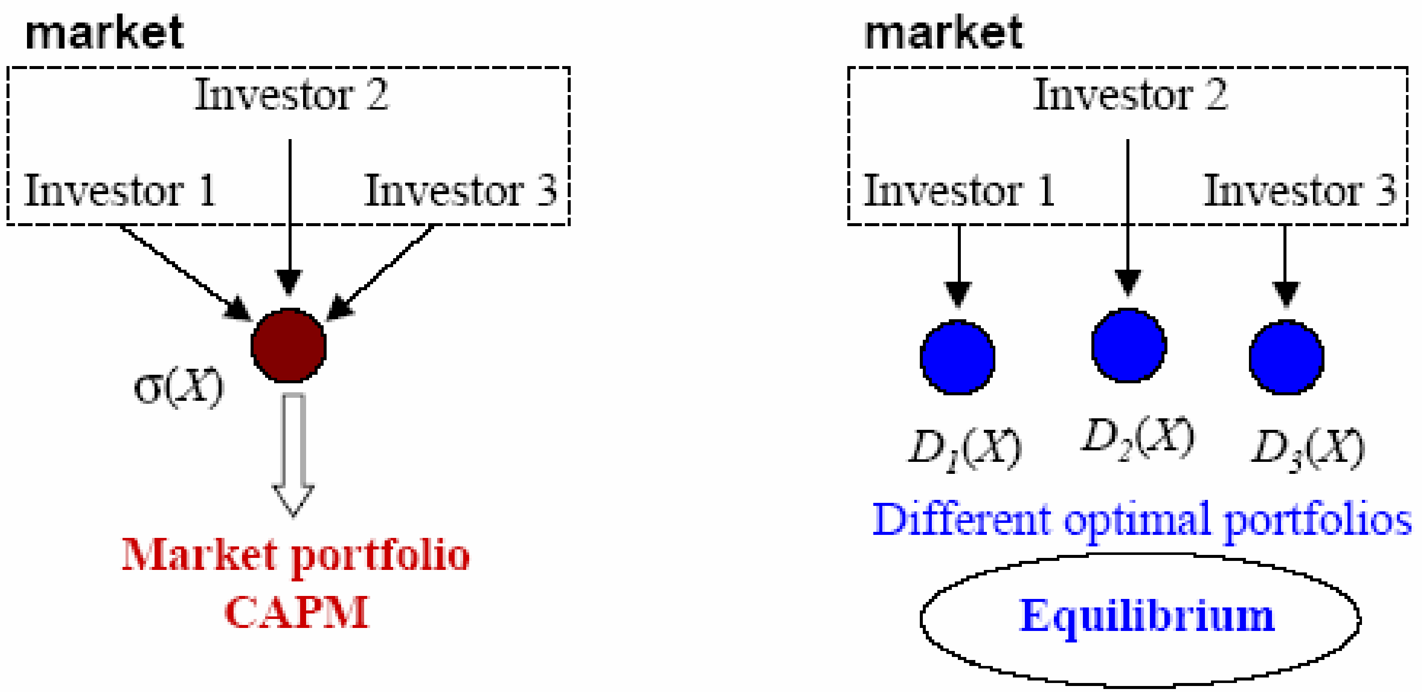
$$\pi_i = \frac{1}{1 + r_0} \left(\underbrace{E\varsigma_i + \frac{Er_M - r_0}{\mathcal{D}(r_M)} \operatorname{cov}\left(\varsigma_i, G\right)}_{\text{certainty equivalent}} \right)$$

Market Equilibrium and CAPM

Classical covariance relations: one-factor predictive model (CAPM)

$$r_i - r_0 \approx \beta_i[r_M - r_0] \text{ for } i = 1, \ldots, n.$$

- β_i may not be uniquely determined
- r_M nonuniqueness: “flat spot” on the efficient frontier



Equilibrium with Several Deviation Measures

Assumptions:

• Several groups of investors

• Each group has the same utility function $U_i(ER_i, \mathcal{D}_i(R_i))$ based on deviation measure $\mathcal{D}_i$.

• Utility functions
  depend on mean and deviation
  concave w.r.t. mean and deviation
  increasing w.r.t. mean and decreasing w.r.t. deviation

• Investors maximize their utility functions subject to budget constraint

Equilibrium results:

1. An equilibrium exists with respect to deviation measures $\mathcal{D}_i$
2. Each group of investors has its own masterfund
3. Investors invest in risk-free asset and their own masterfunds

Optimality Conditions and CAPM

Assumption: low risk-free rate, $r_0 < r_0^-$

- **Capital Asset Pricing Model (CAPM)**

$$\bar{r}_i - r_0 = \beta_i(\bar{r}_M - r_0), \qquad \beta_i = \frac{\mathrm{covar}(G, r_i)}{\mathcal{D}(r_M)}$$

G = risk identifier for the market fund r_M

CAPM: Standard Deviation

- **Asset's** β_i

$$\beta_i = \frac{\mathrm{covar}(r_i, r_M)}{\sigma^2(r_M)}$$

CAPM-like Relations: Semideviation

Lower semideviation

$$\sigma_{-}(X) = (E[\max\{EX - X, 0\}^2])^{1/2}$$

- **CAPM-like relation**

$$\beta_i = \frac{\mathrm{covar}(r_i, r_M^-)}{\sigma_-^2(r_M)}$$

CAPM-like Relations: Range

Deviation measure from range, $EX - \inf X$

- If $X^*(\omega) = \inf X^*$ for unique ω^* then Q^* is uniquely determined

CAPM-like relation

$$\beta_i = \frac{\bar{r}_i - E\left[r_i \,\middle|\, r_M = \inf r_M\right]}{\bar{r}_M - \inf r_M}$$

CAPM-like Relations: GMAD

Generalized Mean Absolute Deviation

$$\mathcal{D}(X) = E\big[\, a(\omega)|X(\omega) - EX|\, \big]$$

- If $r = (r_1, \ldots, r_n)$ continuously distributed, then

$$\beta_{\mathrm{i}} = \frac{\mathrm{E}\big[\, \mathrm{a}(\mathrm{r_i} - \bar{\mathrm{r}}_{\mathrm{i}})\, \mathrm{sign}(\mathrm{r_M} - \bar{\mathrm{r}}_{\mathrm{M}})\, \big]}{\mathrm{E}\big[\, \mathrm{a}|\mathrm{r_M} - \bar{\mathrm{r}}_{\mathrm{M}}|\, \big]}$$

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Rockafellar, R. T., Uryasev, S. and M. Zabarankin: **Deviation Measures in Generalized Linear Regression**. Research Report 2002-9. ISE Dept., University of Florida, December 2002

Generalized Regression Problem

Approximate random variable Y in terms of random variables $X_1, X_2, \ldots, X_n$.

$$\min \ \mathcal{E}\left(Y - [c_0 + c_1 X_1 + \ldots + c_n X_n]\right)$$

Error measure $\mathcal{E}$ satisfies axioms:

(E1) $\mathcal{E}(0) = 0,\ \mathcal{E}(X) > 0,\ \text{for } X \neq 0,$

(E2) $\mathcal{E}(\lambda X) = \lambda \mathcal{E}(X) \text{ for } \lambda \geq 0,$

(E3) $\mathcal{E}(X + X') \leq \mathcal{E}(X) + \mathcal{E}(X'),$

(E3) Lower semicontinuity

Error, Deviation, Statistic

For an error measure $\mathcal{E}$,

the corresponding deviation measure $\mathcal{D}$ is

$$\mathcal{D}(X) = \min_{C} \mathcal{E}(X - C)$$

the corresponding statistic $\mathcal{S}$ is

$$\mathcal{S}(X) = \arg\min_{C} \mathcal{E}(X - C)$$

Theorem 1: Error Measure for Mixed Deviations

Let $(\mathcal{E}_k, \mathcal{S}_k, \mathcal{D}_k)$, *k=1,...,r*, be a collection of triplets (error, deviation, statistic).

Consider a mixed-deviation measure $\mathcal{D} = \sum_{k=1}^{r} \lambda_k \mathcal{D}_k, \ \lambda_k > 0, \ \sum_{k=1}^{r} \lambda_k = 1.$

The corresponding error measure is defines through

$$\mathcal{E}(X) = \min \left\{ \sum_{k=1}^{r} \lambda_k \mathcal{E}_k (X - C_k) \,\middle|\, C_1, \ldots, C_r \text{ such that } \sum_{k=1}^{r} \lambda_k C_k = 0 \right\}$$

The corresponding statistic is

$$\mathcal{S}(X) = \sum_{k=1}^{r} \lambda_k \mathcal{S}_k (X) = \left\{ \sum_{k=1}^{r} \lambda_k C_k \,\middle|\, C_k \in \mathcal{S}_k (X) \right\}$$

Theorem 2: Separation Principle

General regression problem

$$\min \ \mathcal{E}\left(Y - [c_0 + c_1 X_1 + \ldots + c_n X_n]\right)$$

is equivalent to

$$\min \ \mathcal{D}\left(Y - [c_1 X_1 + \ldots + c_n X_n]\right)$$

$$c_0 \in \mathcal{S}(Y - [c_1 X_1 + \ldots + c_n X_n])$$

Example 1: CVaR

Error: $\mathcal{E}_{KB}^{\alpha}(X) = E[X_+] + (\alpha^{-1} - 1)E[X_-]$

Deviation: $\mathcal{D}(X) = CVaR_\alpha(X - EX)$

Risk: $\mathcal{R}(X) = CVaR_\alpha(X)$

Statistic: $\mathcal{S}(X) = -VaR_\alpha(X)$

Koenker, R., Bassett, G. Regression quantiles. Econometrica 46, 33–50 (1978)

$$\min_{C \in R} \left(E[X - C]_+ + (\alpha^{-1} - 1)E[X - C]_- \right) = CVaR_\alpha(X - EX)$$

$$\arg\min_{C \in R} \left(E[X - C]_+ + (\alpha^{-1} - 1)E[X - C]_- \right) = -VaR_\alpha(X)$$

Example 2: Mixed-CVaR

Error: $\mathcal{E}(X) = \sum_{k=1}^{r} \lambda_k \mathcal{E}_{KB}^{\alpha_k}(X)$

Deviation: $\mathcal{D}(X) = \sum_{k=1}^{r} \lambda_k CVaR_{\alpha_k}(X - EX)$

Risk: $\mathcal{R}(X) = \sum_{k=1}^{r} \lambda_k CVaR_{\alpha_k}(X)$

Statistic: $\mathcal{S}(X) = -\sum_{k=1}^{r} \lambda_k VaR_{\alpha_k}(X)$

$$\lambda_k > 0, \sum_{k=1}^{r} \lambda_k = 1$$

$$\min\left\{ \sum_{k=1}^{r} \lambda_k \mathcal{E}_{KB}^{\alpha_k}(X - C - C_k) \,\Big|\, C_1,\ldots,C_r \text{ such that } \sum_{k=1}^{r} \lambda_k C_k = 0 \right\} = \sum_{k=1}^{r} \lambda_k CVaR_{\alpha_k}(X - EX)$$

$$\arg\min_{C}\left\{ \sum_{k=1}^{r} \lambda_k \mathcal{E}_{KB}^{\alpha_k}(X - C - C_k) \,\Big|\, C_1,\ldots,C_r \text{ such that } \sum_{k=1}^{r} \lambda_k C_k = 0 \right\} = -\sum_{k=1}^{r} \lambda_k VaR_{\alpha_k}(X)$$

Closing Remarks

Summary:

- Recently Risk Management theory made a significant progress
- Relationship between error, deviation, risk, and risk identifiers
- Major parts of financial theory are developed for general deviation measures
  - CAPM
  - Asset betas
  - One fund theorem
  - Two find theorem
  - Equilibrium with diversity of risk preferences
- Beautiful interplay of optimization, risk management, statistics, and applications (finance, military, material science)

Closing remarks

Some open issues:

- **Relations between risk measures and statistics**

- **Practical applications of the theory:**

  - Restoring risk preferences of investors in terms of deviation measures

  - Empirical testing of the generalized CAPM